

TOWARDS LOWER BOUNDS FOR COMPLEXITY OF 3-MANIFOLDS: A PROGRAM

S. ANISOV

ABSTRACT. For a 3-dimensional manifold M^3 , its complexity $c(M^3)$, introduced by S. Matveev, is the minimal number of vertices of an almost simple spine of M^3 ; in many cases it is equal to the minimal number of tetrahedra in a singular triangulation of M^3 . An approach to estimating $c(M^3)$ from below for total spaces of torus bundles over S^1 , based on the study of θ -curves in the fibers, is developed, and pseudominimal special spines for these manifolds are constructed, which we conjecture to be their minimal spines. We also show how to apply some of these ideas to other 3-manifolds.

CONTENTS

§1. Introduction	
1.1. Brief outlook	2
1.2. Definitions	3
1.3. Example: lens spaces	5
§2. Upper bound	
2.1. θ -curves	7
2.2. On $\mathrm{SL}(2, \mathbb{Z})$, $c(A)$, and $c(\mathcal{A})$	9
2.3. Spines of torus bundles	14
2.4. Digression: spines of lens spaces	20
§3. Lower bound for C^1 -smooth spines transversal to fibers	
3.1. Morse transformations in simple polyhedra	23
3.2. θ -curves in the fibers	26
3.3. Proof of Theorem 12	30
§4. Main conjectures	
4.1. Triangulations of T^2 coming from spines	31
4.2. Can hyperbolic geometry help?	37
4.3. Other 3-dimensional manifolds	40
References	42

§1. INTRODUCTION

1.1. Brief outlook.

The notion of *complexity* of three-dimensional manifolds was introduced by S. Matveev in 1990, see [15]. This complexity is a natural “filtration” on the set of compact 3-manifolds. It is additive with respect to taking the connected sum of manifolds, and for any $k \in \mathbb{Z}$ there are only finitely many compact prime 3-manifolds of complexity at most k ; they can be enumerated by a simple algorithm (however, most of them appear many times in the list obtained, for example, in the list in [14, §5.2]). For any compact prime 3-manifold M different from S^3 , \mathbb{RP}^3 , $L_{3,1}$, and $S^2 \times S^1$, the complexity $c(M)$ is nothing but the minimal possible number of tetrahedra in a singular triangulation of M . On the other hand, these four manifolds are the only closed prime manifolds of complexity 0.

The problem of evaluating the complexity of 3-manifolds is unresolved and appears to be very difficult. The only manifolds of known complexity are those with complexity less than or equal to 7. Their lists in [14] and [19] are obtained by enumeration of all special spines of closed orientable 3-manifolds of small complexity (by the algorithm mentioned above), followed by determining which of the spines obtained are equivalent (that is, are spines of the same manifold). This “equivalence problem” is difficult.

Obviously, any almost simple spine (or singular triangulation) of a manifold M provides an upper bound for $c(M)$. There is an algorithm for simplification of a given spine, see [13]; for all manifolds from [14] and [19], this algorithm is efficient, that is, stops at a minimal spine of a manifold. There is no proof of efficiency of this algorithm in the general case, although one can, of course, use it to find quite reasonable upper bounds for $c(M)$.

Much less is known about lower bounds. Clearly, $c(M) > 7$ whenever M is not homeomorphic to any manifold from tables in [14, 19]. Also, one can easily show that $c(M) \geq b_1(M, G) - 1$ for any commutative group G ; here b_1 is the first Betti number. However, in most cases these estimates are very inadequate. Up to now, the only known way to prove that $c(M) = k$ for some $k > 0$, where M is a closed prime three-manifold, is to construct a special spine of M with k vertices (or a singular triangulation of M with k tetrahedra) and verify that M is not homeomorphic to any manifold of lower complexity.

Here we study 3-manifolds that can be fibered over the circle with torus fiber. We present “reasonable” special spines of these manifolds and discuss different ways of obtaining lower bounds for their complexity. The conjecture that arised is very similar to S. Matveev’s conjecture about the complexity of the lens spaces.

The paper is organized as follows: in the rest of §1, we give necessary definitions (following mainly [14]) and discuss Matveev’s conjecture on the complexity of lens spaces, which implies, by the way, an unexpected theorem on the running time of the Euclid algorithm (Theorem 3 in §1.3). In §2.3, we construct pseudominimal spines with small number of vertices for the total spaces of torus bundles over the circle; as a byproduct, in §2.4 we obtain another description of pseudominimal spines of lens spaces. Both constructions are based on the study of θ -curves in tori, $\mathrm{SL}(2, \mathbb{Z})$ -action on the set Σ of their isotopy classes, flips, and the distance d on Σ defined by flips, see §2.1 and §2.2; another point of view on the flip-distance is presented in §4.2.

We conjecture that the spines constructed in §2.3 and §2.4 are minimal, that is,

the complexity of the corresponding manifolds is equal to the number of vertices in those spines, and the rest of the paper is devoted to this conjecture. In §3 we restrict ourselves to the spines of these spaces that are transversal to the fibers and prove that the number of vertices of any special spine with this property is at least one fifth of its conjectured value, see Theorem 12 (which holds for all Stallings manifolds, not for torus bundles only, see §4.3.2). Till the end of §3, everything relies heavily on the study of θ -curves, which form a metric space (Σ, d) . A generalization of this construction is presented in §4; potentially, it can give a good estimate or even the exact value of $c(M)$, but requires to solve a problem about 2-chains similar to the problem discussed in [26, §4]; also see [10]. Finally, in §4.3 we discuss 3-manifolds different from the total spaces of torus bundles.

Apart from study of θ -curves, another approach deserves to be mentioned. For a 3-manifold M , consider its Turaev–Viro invariants, see [28]. The examples of these invariants constructed in [28] involve a root of unity q , $q^n = 1$, of arbitrary degree n . By construction, $TV_q(M)$ is a certain sum of products over all vertices of P of some polynomial expressions in q assigned to the vertices of P , where P is a special spine of M^3 . Obviously, the degree of $TV_q(M)$ in q (or in $\sigma = q + q^{-1}$) does not exceed the maximal degree $d(q)$ (which depends on q only) of an expression corresponding to a vertex of P times the number of vertices of P , where a spine P can be assumed to have the minimal possible number of vertices. So it is possible, in principle, to estimate the complexity of M by dividing the degree of $TV_q(M)$ (in q or in σ) over $d(q)$; since q is an n th root of unity, it makes sense to consider the limit of that ratio as n tends to infinity. We do not elaborate this idea in the present paper; note that evaluating the invariants $TV_q(M)$ is a problem difficult enough by itself, and spines constructed in §2.3 can be used in the calculations.

Acknowledgements. The idea to relate θ -curves and their flips (studied in [2]) to spines of fibered spaces and their complexity is due to V. Turaev (see Fig. 10). R. Fernandes suggested a simplification of the original proof of Theorem 3. The author would also like to thank Yu. Baryshnikov, Yu. Burman, V. Nikulin, M. Polyak, V. Vassiliev, and especially S. Matveev for many useful discussions. The author thanks Instituto Superior Técnico (Lisbon, Portugal), Isaac Newton Institute (Cambridge, U.K.), and IHES (Bures-sur-Yvette, France) for their kind hospitality, and the former Institute for the support of S. Matveev’s visit, which was very important for this work.

1.2. Definitions.

In this section we follow [14, 15]. By K denote the 1-dimensional skeleton of the tetrahedron, which is nothing but the clique (that is, the total graph) with 4 vertices. Note that K is homeomorphic to a circle with three radii.

Definition 1. A compact 2-dimensional polyhedron is called *almost simple* if the link of each of its points can be embedded in K . An almost simple polyhedron P is said to be *simple* if the link of each point of P is homeomorphic to either a circle or a circle with a diameter or the whole graph K . A point of an almost simple polyhedron is *non-singular* if its link is homeomorphic to a circle, it is said to be a *triple point* if its link is homeomorphic to a circle with a diameter, and it is called a *vertex* if its link is homeomorphic to K . The set of singular points of a simple polyhedron P (i.e., the union of the vertices and the triple lines) is called its *singular graph* and is denoted by SP .

It is easy to see that any compact subpolyhedron of an almost simple polyhedron

is almost simple as well. Neighborhoods of non-singular and triple points of a simple polyhedron are shown in Fig. 1 a, b; Fig. 1 c–f represent four equivalent ways of looking at vertices.

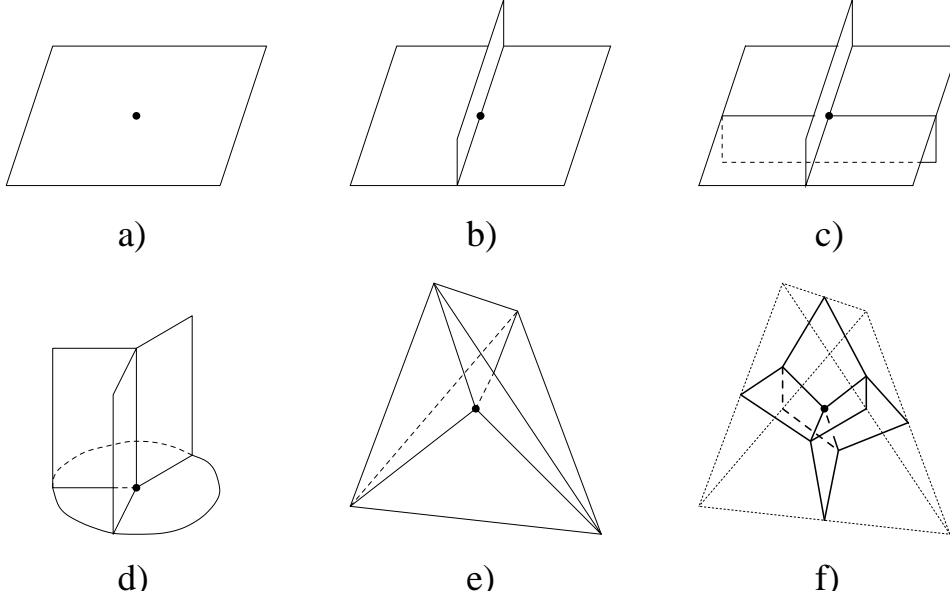


FIGURE 1. Nonsingular (a) and triple (b) points; ways of looking at vertices (c–f)

Definition 2. A simple polyhedron P with at least one vertex is said to be *special* if it contains no closed triple lines (without vertices) and every connected component of $P \setminus SP$ is a 2-dimensional cell.

Definition 3. A polyhedron $P \subset \text{Int } M$ is called a *spine* of a compact 3-dimensional manifold M if $M \setminus P$ is homeomorphic to $\partial M \times (0, 1]$ if $\partial M \neq 0$ or to an open 3-cell if $\partial M = 0$. In the other words, P is a spine of M if a manifold M with boundary (or punctured at one point closed manifold M) can be collapsed onto P . A spine P of a 3-manifold M is said to be *almost simple*, *simple*, or *special* if it is an almost simple, simple, or special polyhedron, respectively.

Definition 4. The *complexity* $c(M)$ of a compact 3-manifold M is the minimal possible number of vertices of an almost simple spine of M . An almost simple spine with the minimal possible number of vertices is said to be a *minimal* spine.

Theorem 1 [7]. *Any compact 3-manifold has a special spine.*

Theorem 2 [15]. *Let M be a compact orientable prime 3-manifold with incompressible (or empty) boundary and without essential annuli. If $c(M) > 0$ (that is, if M is different from (possibly punctured) S^3 , $\mathbb{R}P^3$, $L_{3,1}$, and $S^2 \times S^1$), then any minimal almost simple spine of M is special.*

Recall that a 3-manifold M is called prime if it cannot be represented as a connected sum $M = M_1 \# M_2$ with M_1, M_2 both different from S^3 .

Remark 1. In this paper, we consider lens spaces $L_{p,q}$, $q > 3$, and the total spaces of torus bundles over the circle. All these manifolds satisfy the assumptions of Theorem 2.

Remark 2. Starting from a special spine P of a manifold M , one can triangulate M into n tetrahedra, where n is the number of vertices of P . This singular triangulation has the only vertex somewhere inside $M \setminus P$, its edges are dual to the 2-cells of P , and triangles are dual to the edges of P . On the other hand, given a singular triangulation of M containing n tetrahedra, one can easily obtain a special spine of the manifold M punctured at all vertices of the triangulation. It was shown in [15] that puncturing does not affect the complexity. Thus for a manifold M satisfying assumptions of Theorem 2 (in particular, for any prime manifold without boundary), its complexity $c(M)$ is equal to the minimal possible number of tetrahedra in a singular triangulation of M , provided that $c(M) > 0$.

Remark 3. Let a special spine P of a manifold M without boundary have n vertices. Since each vertex of the graph SP has degree 4, P contains $2n$ edges. Since the Euler characteristic of any 3-manifold equals zero and $M \setminus P$ is a 3-cell, we have the equality $n - 2n + f - 1 = 0$, which implies $f = n + 1$, where f stands for the number of 2-dimensional “faces” of P . It follows from the construction of Remark 2 that the groups $\pi_1(M)$ and $H_1(M)$ have at most f generators. Therefore, $f - 1 = c(M) \geq b_1(M) - 1$.

1.3. Example: lens spaces.

Definition 5. Let p, q be coprime positive integers. The *Euclid complexity* $E(p, q)$ is the number of subtractions (not divisions!) that the Euclid algorithm takes to convert the pair (p, q) into the pair $(0, 1)$. It is easy to see that $E(p, q)$ equals the sum of the denominators of the continued fraction representing any of the rational numbers p/q and q/p .

A good exposition of the Euclid algorithm and continued fraction theory can be found in [7, 29].

Conjecture 1 [14, 15]. *The complexity of the lens space $L_{p,q}$ is equal to $c(L_{p,q}) = E(p, q) - 3$.*

Pseudominimal special spines of the spaces $L_{p,q}$ with $E(p, q) - 3$ vertices were constructed in [13, 14]. Pseudominimality of a spine means that no simplification move can be applied to it; for exact definitions, see [14]. In the end of §2 we present another construction of these spines. Note that the manifolds $L_{p,q}$ and $L_{p,p-q}$ are homeomorphic, and so are the manifolds $L_{p,q}$ and $L_{p,r}$, where $0 < q, r < p$ and $qr \equiv 1 \pmod{p}$. So Conjecture 1 implies that $E(p, q) = E(p, p - q)$ and $E(p, q) = E(p, r)$ for p, q, r as above; if these corollaries did not hold, Conjecture 1 would fail automatically. However, they are true. Indeed, $E(p, q) = E(q, p - q) + 1$ and $E(p, p - q) = E(p - q, q) + 1$, which implies $E(p, q) = E(p, p - q)$. The second corollary is a true statement, too, but this is far less obvious.

Theorem 3. *Let $0 < q, r < p$ and $qr \equiv \pm 1 \pmod{p}$. Then $E(p, q) = E(p, r)$.*

Proof. We can suppose that $p \geq 3$. Let us introduce two *row transformation* matrices

$$R_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Obviously, we have

$$R_1^{\pm 1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \pm c & b \pm d \\ c & d \end{pmatrix} \quad \text{and} \quad R_2^{\pm 1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ a \pm c & b \pm d \end{pmatrix}.$$

Consider the expansion of p/q in a continued fraction

$$\frac{p}{q} = n_1 + \frac{1}{n_2 + \frac{1}{\ddots + \frac{1}{n_k}}}.$$

Set $U = R_2^{-1} R_\varepsilon^{-n_k+1} \dots R_1^{-n_3} R_2^{-n_2} R_1^{-n_1}$, where $\varepsilon = 1$ for k odd and $\varepsilon = 2$ for k even; note that $n_1 \geq 1$ and $n_k \geq 2$. It is easy to see that U takes vector $(p, q)^T$ to $(1, 0)^T$ (where T stands for transposing): $R_1^{-n_1}$ takes $(p, q)^T$ to $(p - n_1 q, q)^T$ and so forth, according to the Euclid algorithm, the only exception being that at the last step we apply R_2^{-1} to $(1, 1)^T$, not R_1^{-1} , in order to convert $(1, 1)^T$ to $(1, 0)^T$, not to $(0, 1)^T$.

First suppose that $qr \equiv -1 \pmod{p}$, thus $qr = sp - 1$ for some positive integer s . Since $1 \leq q < p$ and $1 \leq r < p$, we have $s \leq q$ and $s \leq r$. Let us consider the inverse matrix

$$U^{-1} = R_1^{n_1} R_2^{n_2} R_1^{n_3} \dots R_\varepsilon^{n_k-1} R_2. \quad (1)$$

We proclaim that

$$U^{-1} = \begin{pmatrix} p & r \\ q & s \end{pmatrix}.$$

Indeed, equation (1) implies that U^{-1} has the following properties:

- 1) the first column of U^{-1} is $(p, q)^T$;
- 2) the determinant of U^{-1} equals 1;
- 3) the second column entries of U^{-1} are positive, and
- 4) they do not exceed the corresponding first column entries.

Property 1 follows from the construction of the sequence involved in (1), and property 2 is obvious. It is clear that both second column entries of U^{-1} are nonnegative; they are both positive, because the right hand side of (1) contains both R_1 and R_2 . The last property holds for R_2 and survives under left multiplications by R_1 and R_2 . The first two properties imply that the second column of U^{-1} is $(r + mp, s + mq)^T$ for some $s \in \mathbb{Z}$, and the last two properties show that in fact $m = 0$.

Note that $E(p, q) = n_1 + n_2 + \dots + n_k$ equals the sum of the exponents in (1). Since $R_1^T = R_2$ and $R_2^T = R_1$, for the transposed inverse matrix $(U^{-1})^T$ we have

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = (U^{-1})^T = R_1 R_{3-\varepsilon}^{n_k-1} \dots R_2^{n_3} R_1^{n_2} R_2^{n_1}, \quad (2)$$

where $n_k \geq 2$ and $n_1 \geq 1$. This yields the sequence of $n_1 + n_2 + \dots + n_k = E(p, q)$ subtractions that converts the pair (p, r) into the pair $(1, 0)$. Such a sequence is unique up to a possible application of several subtractions R_1 to the column $(1, 0)^T$ (clearly, R_1 does not change it); it is nothing but the Euclid algorithm provided that there are no those “fake” subtractions. This condition is satisfied, because the last subtraction (the inversed rightmost one in (2), or the leftmost one in a similar expression for U^T) is R_2 , not R_1 . Therefore $E(p, q) = E(p, r)$ for $qr \equiv -1 \pmod{p}$. In the case $qr \equiv 1 \pmod{p}$, note that $q(p - r) \equiv -1 \pmod{p}$ and $0 < p - r < p$. Consequently, $E(p, q) = E(p, p - r) = E(p, r)$. \square

Theorem 3 means that Conjecture 1 passes a nontrivial “sanity test”.

§2. UPPER BOUND

2.1. θ -curves.

Let us recall some definitions and results from [2]¹.

Definition 6. A θ -curve $L \subset T^2$ is a graph with two vertices and three edges (not loops) connecting these vertices, embedded in T^2 in such a way that the edges are pairwise non-homotopic; this is equivalent to the condition that the complement $T^2 \setminus L$ is a 2-dimensional cell.

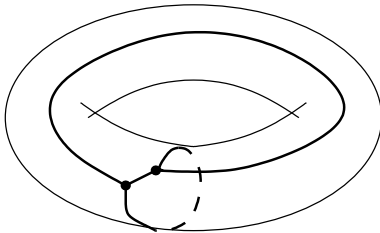


FIGURE 2. A θ -curve

Up to an isotopy, any two θ -curves can be taken to one another by a linear automorphism of the torus, see [2]. Another way to change the isotopy class of a θ -curve is to apply a sequence of flips.

Definition 7. A *flip* along an edge of a trivalent graph (in particular, of a θ -curve) is an invertible restructuring of the graph that acts on a neighborhood of this edge as shown on Fig. 3. A flip does not change the graph outside of this neighborhood.

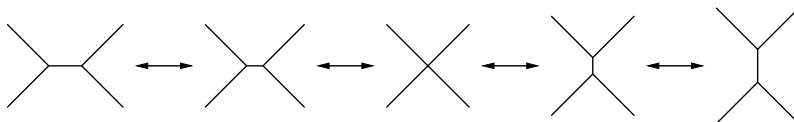


FIGURE 3. A flip

For any two θ -curves L_1, L_2 , there exists a sequence of flips (and isotopies) that takes L_1 to L_2 , see [1, 2]. Now let us recall how one can find the minimal number of flips required for such a sequence.

For a θ -curve L , there are three unoriented (or six oriented) cycles in $\pi_1(T^2)$ formed by pairs of edges of L . These cycles also can be represented by the three 1-cells of the singular triangulation of T^2 dual to the cell decomposition defined by L . The six points in the lattice $\mathbb{Z}^2 = \pi_1(T^2) = H_1(T^2, \mathbb{Z})$ corresponding to these cycles are the vertices of some convex centrally symmetric hexagon $W(L)$.

Definition 8. The hexagon $W(L)$ is said to be *associated* to a θ -curve L . A hexagon W with the vertices in \mathbb{Z}^2 is called *admissible* if it is associated to some θ -curve. The *standard hexagon* is the hexagon W_0 with the vertices $\pm(1, 0)$, $\pm(0, 1)$, and $\pm(1, -1)$, see Fig. 4. It is associated to the θ -curve shown on Fig. 2 (under a natural choice of a parallel and a meridian of T^2 as a basis of $H_1(T^2) = \mathbb{Z}^2$).

¹We do not follow the notation of [2] here.

Theorem 4 [2].

1) A hexagon W (with vertices at lattice points), centrally symmetric with respect to the origin O , is admissible if and only if it has the following properties:

- a) if X and Y are nonopposite vertices of W , then the area of the triangle OXY equals $1/2$;
- b) if X, Y , and Z are three consecutive vertices of W , then $\overrightarrow{OY} = \overrightarrow{OX} + \overrightarrow{OZ}$.

Properties a) and b) are equivalent. The origin is the only interior lattice point of an admissible hexagon W . The vertices of W are the only lattice points on its boundary.

2) Two θ -curves L and L' are isotopic if and only if $W(L) = W(L')$. For any two θ -curves L and L' there exists an operator $A \in \text{SL}(2, \mathbb{Z})$ such that AL is isotopic to L' and $A(W(L)) = W(L')$.

3) Let θ -curves L and L' differ by the flip along an edge e . Then associated hexagons W and W' have in common two pairs of opposite vertices that correspond to cycles σ and μ dual to two other edges. The remaining pair of the vertices is $\pm(\sigma + \mu)$ for one of the hexagons W, W' and $\pm(\sigma - \mu)$ for the other one.

Since a θ -curve has three edges, three different flips can be applied. Fig. 4 shows how they change the standard hexagon W_0 . The result of a flip transformation of an arbitrary admissible hexagon can be represented by the same picture with another coordinate system, because any admissible hexagon is $\text{SL}(2, \mathbb{Z})$ -equivalent to W_0 .

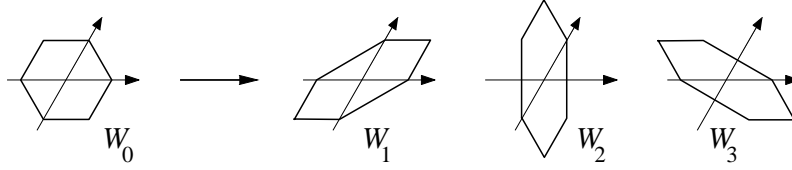


FIGURE 4. Action of three flips on the standard hexagon

According to the second part of Theorem 4, we can study sequences of flips that convert an admissible hexagon W_1 into another admissible hexagon W_2 rather than sequences of flips that take a θ -curve L_1 into another θ -curve L_2 . So it is natural to introduce a graph Γ that has the admissible hexagons as its vertices and flips as its edges; the number of flips required to convert W_1 into W_2 equals the distance between the corresponding vertices of Γ . Clearly, Γ is a trivalent graph. It turns out that Γ is a tree, see [2]. More information about Γ can be found at [25, Ch. II, §1].

Definition 9. A *leading vertex* of an admissible hexagon W is its vertex that is the most distant from the origin with respect to the quadratic form $Q(x, y) = x^2 + xy + y^2$.

The standard hexagon W_0 is a unit regular hexagon with respect to $Q(x, y)$. Any other admissible hexagon has only one pair of opposite leading vertices.

Theorem 5 [2]. Let (p, q) be a leading vertex of an admissible hexagon $W \neq W_0$. Suppose that $p > 0$ and $q > 0$. Then $d(W, W_0) = E(p, q)$, where $d(W, W_0)$ stands for the distance between W and W_0 in Γ and $E(p, q)$ is the Euclid complexity, see Definition 5 above.

In fact, the steps (flips) of the only way from (the vertex of Γ corresponding to) W_0 to (the vertex corresponding to) W in Γ are in a natural one-to-one correspondence with the steps (subtractions) of the Euclid algorithm applied to the pair (p, q) . There is an algorithm that constructs the path from W to W_0 : start at W and apply the flip that decreases the length of a hexagon; such a flip is unique unless the hexagon is W_0 . The numbers p, q are coprime by virtue of the first part of Theorem 4, and for any pair of coprime numbers (p, q) , there exists exactly one admissible hexagon with a leading vertex at (p, q) , cf. [2]. For the detailed proof of Theorem 5, see [2].

If the coordinates (p, q) of a leading vertex of W are both negative, one should consider the opposite vertex, or rotate the coordinate system by π . If $pq < 0$, one should rotate the coordinate system by $\pm\pi/3$ (we consider “triangular” coordinates shown on Fig. 4 rather than rectangular coordinates), that is, to apply the coordinate change $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ or its inverse, to make both coordinates of one of the leading vertices positive. Then Theorem 5 may be applied. This gives the following simple answer: $d(W, W_0) = E(|p|, |q|) - 1$ whenever $pq < 0$, where (p, q) is a leading vertex of W . The -1 summand can be explained as follows: if, for example, $p < 0$ and $q > 0$, the process of converting W into W_0 by flips corresponds to the converting of the unordered pair $(-p, q)$ to the pair $(1, 1)$, not to $(0, 1)$, by subtractions according to the Euclid algorithm (since we can consider $(-1, 1)$ as a leading vertex of W_0); of course, this takes one subtraction less. Also see §4.2.

Definition 10. For a matrix $A \in \text{SL}(2, \mathbb{Z})$ we define its *complexity* $c(A)$ by putting $c(A) = d(W_0, AW_0)$.

To calculate the number $c(A)$, find a leading vertex (p, q) of the hexagon AW_0 . Then we get $c(A) = E(|p|, |q|)$ if $pq > 0$ or $c(A) = E(|p|, |q|) - 1$ if $pq < 0$. In particular, if $AW_0 = W_0$ (there are six matrices A with this property), we get $c(A) = 0$.

2.2. On $\text{SL}(2, \mathbb{Z})$, $c(A)$, and $c(\mathcal{A})$.

Since all admissible hexagons are $\text{SL}(2, \mathbb{Z})$ -equivalent and the action of this group preserves the existence of a flip connecting two given hexagons (thus, there is an action of $\text{SL}(2, \mathbb{Z})$ on the graph Γ), we have $d(W_1, W_2) = d(BW_1, BW_2)$ for $B \in \text{SL}(2, \mathbb{Z})$. In particular,

$$c(A^{-1}) = d(W_0, A^{-1}W_0) = d(AW_0, AA^{-1}W_0) = d(W_0, AW_0) = c(A) \quad (3)$$

and $c(A) = d(W_0, AW_0) = d(BW_0, BAW_0)$, which may be different from $d(BW_0, ABW_0) = d(W, AW)$. Thus we have $c(A) = d(W, BAB^{-1}W)$ for any admissible hexagon $W = BW_0$.

It is not true that $c(A) = d(W, AW)$ for any admissible hexagon W . This means that the number $c(A)$ is not a conjugacy class invariant, contrary to a statement contained implicitly in [2].

Example. Let $A = \begin{pmatrix} 171 & 100 \\ -289 & -169 \end{pmatrix}$ and $B = \begin{pmatrix} 10 & -17 \\ -17 & 29 \end{pmatrix}$. Note that $A, B \in \text{SL}(2, \mathbb{Z})$. By a straightforward calculation, we obtain $c(A) = 13$, while for a conjugate matrix $A' = B^{-1}AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ we have $c(A') = 1$.

The following problem arises: find the minimal value of $c(A)$ over the whole conjugacy class of A in $\text{SL}(2, \mathbb{Z})$.

Definition 11. The *complexity* of an operator $\mathcal{A} \in \mathrm{SL}(2, \mathbb{Z})$ is the minimal possible complexity of matrices that represent \mathcal{A} in all possible bases of the lattice \mathbb{Z}^2 :

$$c(\mathcal{A}) = \min_{A \sim A_0} c(A) = \min_{B \in \mathrm{SL}(2, \mathbb{Z})} c(B^{-1}A_0B),$$

where A_0 is any matrix representing \mathcal{A} and \sim denotes conjugacy in $\mathrm{SL}(2, \mathbb{Z})$. In other words, $c(\mathcal{A}) = \min d(W, \mathcal{A}W)$ over all admissible hexagons W . A matrix A of an operator \mathcal{A} in some basis is said to be a *minimal* matrix of \mathcal{A} if $c(A) = c(\mathcal{A})$, that is, if $c(A) \leq c(A')$ for any matrix A' conjugated to A . An admissible hexagon W is called *minimal* for \mathcal{A} if $d(W, \mathcal{A}W) = c(\mathcal{A})$.

In §3, we will be interested in the sequence $\{c(\mathcal{A}^k)\}$. Properties of this sequence depend on the trace of \mathcal{A} . Recall (see [6, §0]) that the operator \mathcal{A} is called *elliptic* if $|\mathrm{tr} \mathcal{A}| < 2$. In this case $\mathrm{tr} \mathcal{A} = 0$ or $\mathrm{tr} \mathcal{A} = \pm 1$, and the equation $\mathcal{A}^2 - (\mathrm{tr} \mathcal{A})\mathcal{A} + (\det \mathcal{A})I = 0$ implies either $\mathcal{A}^2 = -I$ or $\mathcal{A}^3 \pm I = 0$, because $\det \mathcal{A} = 1$. Thus elliptic operators are periodic of period 3, 4 or 6, and so are the sequences $\{c(\mathcal{A}^k)\}$; both eigenvalues of an elliptic operator are roots of unity. If $\mathrm{tr} \mathcal{A} = \pm 2$, we say that \mathcal{A} is a *parabolic* operator. In this case $(\mathcal{A} \pm I)^2 = 0$ and \mathcal{A} is $\mathrm{SL}(2, \mathbb{Z})$ conjugated to either $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & n \\ 0 & -1 \end{pmatrix}$, where $n \in \mathbb{Z}$. So \mathcal{A} is either a periodic operator (if $n = 0$) or, up to a sign, a power of the Jordan block $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; both eigenvalues of a parabolic operator equal ± 1 . Finally, \mathcal{A} is *hyperbolic* if $|\mathrm{tr} \mathcal{A}| > 2$. In this case the eigenvalues of \mathcal{A} are different real numbers, and \mathcal{A} is a hyperbolic rotation. Also see [22, §5].

Lemma 1. $c(AB) \leq c(A) + c(B)$. Moreover, $c(AB) \equiv c(A) + c(B) \pmod{2}$.

Proof. By definition, $c(A) = d(W_0, AW_0)$, $c(AB) = d(W_0, ABW_0)$, and $c(B) = d(W_0, BW_0) = d(AW_0, ABW_0)$. Since d is a metric on Γ , the triangle inequality $d(W_0, ABW_0) \leq d(W_0, AW_0) + d(AW_0, ABW_0)$ holds. Since the graph Γ is a tree, we have $d(W_0, ABW_0) = d(W_0, AW_0) + d(AW_0, ABW_0) - 2k$, where $k \in \mathbb{Z}_{\geq 0}$, see Fig. 5. \square

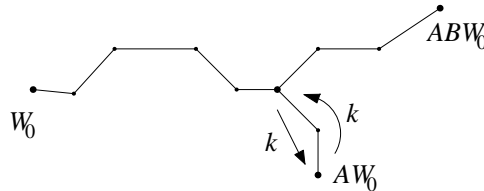


FIGURE 5. Additivity of tree distance parities

Theorem 6. Reduction of $c(A)$ modulo 2 coincides with the unique epimorphism of the modular group $\mathrm{SL}(2, \mathbb{Z})/\{\pm I\}$ to \mathbb{Z}_2 .

Proof. Lemma 1 implies that this reduction is a homomorphism of $\mathrm{SL}(2, \mathbb{Z})$ to \mathbb{Z}_2 . It is an epimorphism since it takes $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ to 1 mod 2. It is an epimorphism of the modular group $G = \mathrm{SL}(2, \mathbb{Z})/\{\pm I\}$, because $c(A) = c(-A)$: the hexagon AW_0 is centrally symmetric, whence $AW_0 = -AW_0$. Such an epimorphism φ is unique.

Indeed, it is defined by its values $\varphi(S)$ and $\varphi(T)$ on the elements $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which generate the modular group, see [24, Ch. VII, §1]. The relation $(ST)^3 = 1$ in G implies that $\varphi(S) = \varphi(T)$ in Z_2 . Since φ is not identically zero, we have $\varphi(S) = \varphi(T) = 1$. \square

Let us explain how to find a minimal matrix of an operator \mathcal{A} . Let A be its matrix in some basis, W_0 be the standard hexagon in this basis, $W = AW_0$, and $W_0, W_1, \dots, W_{c(A)} = W$ be the shortest path from W_0 to W in Γ . Then $W = AW_0, AW_1, \dots, AW_{c(A)} = AW = A^2W_0$ is the shortest path from W to AW in Γ . Both $W_{c(A)-1}$ and AW_1 are neighbors of a trivalent vertex W of Γ . Compare them.

Theorem 7. *Any matrix A with $c(A) \leq 1$ is minimal. A matrix A with $c(A) > 1$ is minimal if and only if $W_{c(A)-1} \neq AW_1$.*

Proof. If $c(A) = 0$, the matrix A is minimal. Let $c(A) = 1$, whence $c(\mathcal{A}) \leq 1$. It follows from eq. (3) and Lemma 1 that $c(B^{-1}AB) \equiv c(A) + 2c(B) \equiv c(A) = 1 \pmod{2}$. So $c(\mathcal{A})$ is odd, thus $c(\mathcal{A}) \neq 0$ and A is minimal.

Suppose that $W_{c(A)-1} = AW_1$. The operator \mathcal{A} takes W_1 to $AW_1 = W_{c(A)-1}$. We have $c(\mathcal{A}) \leq d(W_1, \mathcal{A}W_1) = d(W_1, W_{c(A)-1}) = c(A) - 2$ (unless $c(A) \leq 1$), i. e., the matrix A is not minimal. This proves the “only if” part of the Theorem.

Suppose that the matrix A is not minimal. This implies that the standard hexagon W_0 is not minimal. Let V be a minimal hexagon for the operator \mathcal{A} . By γ_0 denote the path from V to AV in Γ of length $c(\mathcal{A})$. For any $k \in \mathbb{Z}$ let $\gamma_k = A^k\gamma$ be the path from A^kV to $A^{k+1}V$ in Γ (recall that Γ carries an action of $\text{SL}(2, \mathbb{Z})$). Put $\gamma = \bigcup_{k \in \mathbb{Z}} \gamma_k$. Note that any vertex of Γ on γ represents a minimal hexagon, so $W_0 \in \Gamma \setminus \gamma$. We have to consider three cases.

Case 1: $c(\mathcal{A}) > 1$. Then two vertices of γ neighboring with A^kV , $k \in \mathbb{Z}$, are different because of the “only if” statement proven above. So γ is homeomorphic to a line, because two neighbors of any interior vertex of any γ_k are different, too (since γ_k is the shortest path from A^kV to $A^{k+1}V$), and the graph Γ is a tree.

By W_0U_0 denote the shortest path from W_0 to γ (that is, U_0 is the first point of γ belonging to any path from W_0 to a point of γ). Then WU and $AWAU$ are the shortest paths from W and AW to γ , where $W = AW_0$ and $U = AU_0$. Obviously, these paths end at different points of γ : if $U_0 \in \gamma_k$, then $U \in \gamma_{k+1}$ and $AU \in \gamma_{k+2}$. Since Γ is a tree, the paths W_0U_0 , WU , and $AWAU$ are mutually disjoint. This means that W_0U_0UW is the shortest path from W_0 to W and $WUAUAW$ is the shortest path from W to AW , see Fig. 6. The leg $WU = A(W_0U_0)$ of this path is not empty. Consequently, the penultimate vertex of the path W_0W coincides with the first (different from W) vertex of the path WAW , which proves the statement of the “if” part of the Theorem.

Case 2: $c(\mathcal{A}) = 1$. Let V be a minimal hexagon for \mathcal{A} . If $A^2V \neq V$, then $A^{k+2}V \neq A^kV$ for any $k \in \mathbb{Z}$, all the γ_k are different, γ is homeomorphic to a line, and we can repeat the argument of Case 1. If $A^2V = V$, we have $A^kV = V$ for k even and $A^kV = AV$ for k odd. Without loss of generality, it can be assumed that the path from W_0 to V does not pass through AV . The transformation \mathcal{A} takes this path to the path from $W = AW_0$ to AV , which does not pass through V . So the shortest path from W_0 to W is W_0VAVW , and thus it contains the edge VAV .

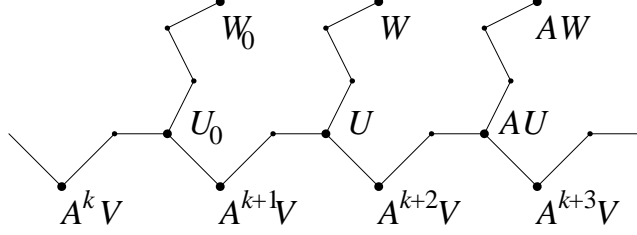


FIGURE 6. Paths $W_0 W$ and $W A W$ overlap

Similarly, the path from W to AW also contains that edge. Therefore, these two paths overlap, which proves the Theorem in the case $c(\mathcal{A}) = 1$.

Case 3: $c(\mathcal{A}) = 0$. There exists an admissible hexagon V such that $AV = V$, but $AW_0 \neq W_0$ because A is not a minimal matrix. Consider paths $W_0 V$ and $W V$ in Γ , where $W = AW_0$. Let U be the first (most distant from V) common point of this paths. Recall that A acts on Γ and takes the path $W_0 V$ to $W V$. Thus $AU = U$, $W_0 \neq U$, and A takes the path $W_0 U$ to $W U$ and the path $W U$ to $AW U$. So the paths $W_0 W = W_0 U W$ and $W A W = W U A W$ overlap over the leg $W U$. \square

Now we can present an algorithm that finds the number $c(\mathcal{A})$ and a minimal matrix A of an operator \mathcal{A} . Start with any matrix A representing this operator. Apply the criterion of Theorem 7. If either $c(A) \leq 1$ or $W_{c(A)-1} \neq AW_1$, the matrix A is minimal. Otherwise, let V be the last common vertex of the paths $W W_0$ and $W A W$. Then V lies on γ and is a minimal hexagon for \mathcal{A} . Choose a basis so that V is the standard hexagon. The matrix of \mathcal{A} in this basis is minimal, and $c(\mathcal{A})$ is equal to complexity of this matrix.

Corollary. *The subgraph $\gamma \subset \Gamma$ constructed in the proof of Theorem 7 is a line if and only if the operator \mathcal{A} is not periodic.* \square

This condition holds if and only if either $c(\mathcal{A}) \geq 2$ or $c(\mathcal{A}) = 1$ and $\mathcal{A}^2 \neq -I$. The line γ is the “mainstream” of the action of \mathcal{A} on Γ . Minimal hexagons for \mathcal{A} are exactly those corresponding to the vertices of the subgraph $\gamma \subset \Gamma$. If γ is a line, there are at most $3c(\mathcal{A})$ different minimal matrices for \mathcal{A} , because any minimal hexagon yields 6 different bases $(OX_1 X_2, OX_2 X_3, \dots, OX_6 X_1)$, where X_1, \dots, X_6 are the vertices of a hexagon and O is the origin), bases that differ by a central symmetry give the same matrix expression of \mathcal{A} , and hexagons V and AV lead to the same set of matrices of \mathcal{A} .

If $c(\mathcal{A}) = 1$, then the minimal matrix for \mathcal{A} is either one of Jordan blocks $\begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}$, $\begin{pmatrix} \pm 1 & -1 \\ 0 & \pm 1 \end{pmatrix}$ or the $\pm\pi/2$ rotation matrix $\begin{pmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{pmatrix}$. These six matrices belong to six different conjugacy classes in $SL(2, \mathbb{Z})$. In the case of a Jordan block, there are three minimal matrices for \mathcal{A} and an infinite number of minimal hexagons (which lie on the line γ). For a rotation, there are three different minimal matrices (namely, $\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in the case of counterclockwise rotation) and only two minimal hexagons, which have in common the four lattice points where a positively definite integer quadratic form $Q_{\mathcal{A}}(\vec{v}) = \det(\vec{v}, \mathcal{A}\vec{v})$ (for the clockwise rotation, we set $Q_{\mathcal{A}}(\vec{v}) = -\det(\vec{v}, \mathcal{A}\vec{v})$) attains its minimal positive value 1.

If $c(\mathcal{A}) = 0$ and $\mathcal{A} \neq \pm I$, the minimal hexagon is unique; its six vertices are the six lattice points where a positively definite integer quadratic form $Q_{\mathcal{A}}(\vec{v}) =$

$\pm \det(\vec{v}, \mathcal{A}\vec{v})$ attains value 1. The minimal matrix for \mathcal{A} is also unique. Finally, for $\mathcal{A} = \pm I$, any admissible hexagon is minimal, while the only minimal matrix is, of course, $\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We omit the proofs of the statements of this paragraph and two preceding ones; most of them are straightforward.

Theorem 8. *Let \mathcal{A} be a non-periodic operator. Then:*

- 1) *for any integer $k \neq 0$, A^k is a minimal matrix for \mathcal{A}^k if and only if A is a minimal matrix for \mathcal{A} ;*
- 2) *$c(\mathcal{A}^k) = |k|c(\mathcal{A})$ for any $k \in \mathbb{Z}$;*
- 3) *for any integer $k \neq 0$, we have $c(A^k) = |k|c(\mathcal{A}) + b$, where $b = c(A) - c(\mathcal{A})$ is a nonnegative even number.*

Proof. It follows from the proof of Theorem 7 and Corollary that the path from W_0 to $A^k W_0$ consists of three legs $W_0 U_0$, $U_0 A^k U_0$, and $A^k U_0 A^k W_0 = A^k(U_0 W_0)$. If the first leg (and, simultaneously, the last one) is empty (contains no edges), matrices A and \mathcal{A}^k are both minimal. Otherwise, neither A nor \mathcal{A}^k is minimal. This proves the first statement and shows that the mainstreams of \mathcal{A} and \mathcal{A}^k coincide: $\gamma(\mathcal{A}^k) = \gamma(\mathcal{A})$. The second statement of the Theorem follows from the first one whenever $k \neq 0$; the case $k = 0$ is trivial. The third statement follows from the description of the path from W_0 to $A^k W_0$ in Γ , see above. \square

We conclude the section on $\text{SL}(2, \mathbb{Z})$, $c(A)$, and $c(\mathcal{A})$ with one more way of looking at the mainstream $\gamma(\mathcal{A})$. Let $\mathcal{A} \in \text{SL}(2, \mathbb{Z})$ be a hyperbolic rotation, so $|\text{tr } \mathcal{A}| > 2$ and eigenvalues of \mathcal{A} are different real numbers. Since admissible hexagons are centrally symmetric, we may assume that the eigenvalues of \mathcal{A} are positive. The eigenvalues and the slopes of eigenvectors are quadratic irrationals, because the discriminant of the characteristic equation $\lambda^2 - (\text{tr } \mathcal{A})\lambda + \det \mathcal{A} = 0$ equals $(\text{tr } \mathcal{A})^2 - 4$ and is not equal to a square of an integer whenever $|\text{tr } \mathcal{A}| > 2$. Draw through the origin two lines l_1, l_2 parallel to eigenvectors. They divide the plane into four parts. For each of these parts, consider the convex hull of the lattice points inside it. Since \mathcal{A} preserves l_i , it preserves the convex hulls h_1, \dots, h_4 , as well as their boundaries, which are infinite sequences of segments. The group of the integers acts on ∂h_i by taking $x \in \partial h_i$ to $\mathcal{A}^k x \in \partial h_i$ for $k \in \mathbb{Z}$. An admissible hexagon is minimal for \mathcal{A} (i.e., belongs to $\gamma(\mathcal{A})$) if and only if its leading vertex (and hence all its other vertices) lies on some ∂h_i . The path from a minimal hexagon $W \in \gamma$ to its image $\mathcal{A}W$ corresponds to the period of the continued fraction expansion of the slope α of l_i , which is a quadratic irrational number. An $\text{SL}(2, \mathbb{Z})$ coordinate change does not affect this period: it takes α to $\frac{a\alpha + b}{c\alpha + d}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$, changing the beginning of the continued fraction expansion of a quadratic irrational number without affecting its periodic part. We leave the proofs of these statements to the reader.

Example. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. This is a minimal matrix of a hyperbolic operator of complexity 2. The boundaries of the convex hulls h_i are represented on Fig. 7 by dotted lines. The coordinates of their corners in this case are, up to signs, the pairs of consecutive Fibonacci numbers. The standard hexagon W_0 , drawn in a bold line, is a minimal hexagon for \mathcal{A} . The hexagons $A^{-1}W_0$ and AW_0 also belong to the mainstream $\gamma(\mathcal{A})$. Since $c(\mathcal{A}) = 2$, there are two orbits of the action of the

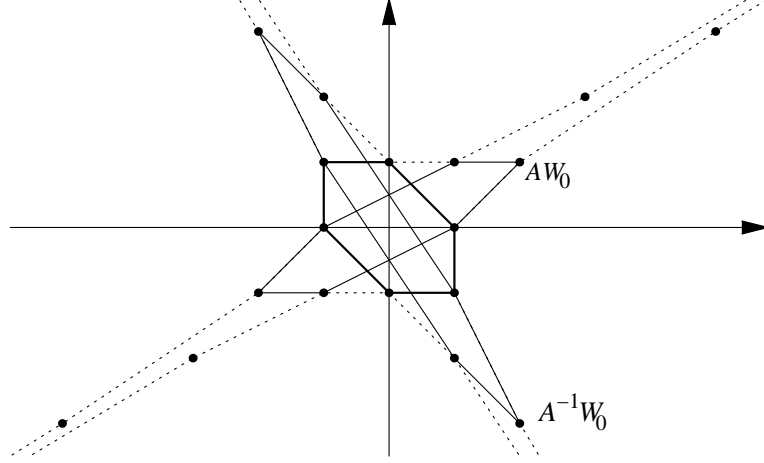


FIGURE 7. The mainstream for a hyperbolic rotation

group \mathbb{Z} on $\gamma(\mathcal{A})$ defined by the rule $k(W) = A^k W$ for $k \in \mathbb{Z}$ and $W \in \gamma(\mathcal{A})$. All three hexagons on Fig. 7 belong to one of the orbits. Hexagons of the other orbit can be obtained from them by the $\pi/2$ rotation around the origin (the eigenvectors of A , which direct l_1 and l_2 , are orthogonal, because $A^T = A$). As $k \rightarrow -\infty$, the hexagon $A^k W$ looks more and more like the line l_1 ; as $k \rightarrow \infty$, it looks more and more like l_2 . Directions of these lines are points of the projective line $\mathbb{RP}^1 = S^1$, which is the absolute in the Poincaré circle model of the Lobachevskii plane. We encourage the reader to find the relation between \mathcal{A} , $\gamma(\mathcal{A})$, and the geodesic line that connects these two absolute points. Hint: see [18, §17] and §4.2 below.

2.3. Spines of torus bundles.

From now on, M denotes the total space of an orientable T^2 -bundle over the circle and $\mathcal{A} \in \text{SL}(2, \mathbb{Z})$ is the monodromy operator (acting on the one-dimensional homology group of the fiber containing the base point of M) of the bundle. By $M(\mathcal{A})$ denote the manifold M corresponding to the monodromy operator \mathcal{A} .

In this section, we construct a pseudominimal (see Definition 12 below) special spine of $M(\mathcal{A})$ with $\max(6, c(\mathcal{A}) + 5)$ vertices. First, let us note that the number $c(\mathcal{A})$ is well defined by the manifold M^3 because of the following statement.

Theorem 9 [27]. *Suppose that 3-manifolds $M(A)$ and $M(B)$, where $A, B \in \text{SL}(2, \mathbb{Z})$, are homeomorphic. Then B is $\text{GL}(2, \mathbb{Z})$ -conjugate to either A or A^{-1} .*

If A and B are $\text{SL}(2, \mathbb{Z})$ -conjugate, they yield the same number $c(\mathcal{A})$. If $B \sim A^{-1}$, then $c(\mathcal{B}) = c(\mathcal{A})$ by virtue of eq. (3). Since the group $\text{GL}(2, \mathbb{Z})$ is generated by its subgroup $\text{SL}(2, \mathbb{Z})$ and the element $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, it now suffices to check that conjugation by C does not affect the complexity $c(A)$; however, this is clear from Definition 10.

Definition 12 [14]. A 2-dimensional component α of a special polyhedron has a *counterpass* if its boundary $\partial\alpha$ passes along some edge of SP in both directions; it is called a *component with short boundary* if $\partial\alpha$ passes through at most 3 vertices and visits any of them only once. A special spine of a 3-dimensional manifold is said to be *pseudominimal* if it contains neither components with counterpasses nor components with short boundaries.

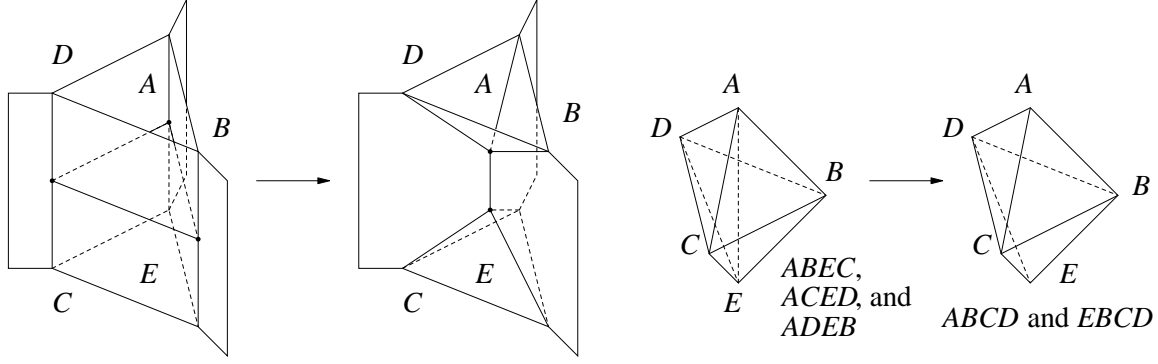


FIGURE 8. A simplification move (left) and the corresponding Pachner move (right)

If a special spine P is not pseudominimal, it is not minimal, because one can apply simplification moves (see [14]) to P and get an almost simple spine with a smaller number of vertices. For example, Figure 8 shows the effect of a simplification move applied to a special spine with a triangular component (the middle horizontal triangle in the left part of Fig. 8); it is easy to see that the neighborhood of a 2-cell with short boundary of length 3 in a special polyhedron P looks like the left hand side of Fig. 8. This move does not change the spine outside of the fragment shown on Fig. 8. Note that the spine obtained is special again: the move produces neither closed triple lines nor non-cellular 2-dimensional components.

Remark. Consider the singular triangulation dual to a special spine with a triangular component. Then the simplification move shown on Fig. 8 corresponds to the three-dimensional $(3, 2)$ Pachner move [20], which replaces three tetrahedra by two tetrahedra. In the two-dimensional case, a flip (see Fig. 3) corresponds to the $(2, 2)$ Pachner move, which switches the diagonal in a quadrilateral formed by two neighboring triangles. Recall that the move shown on Fig. 8 and its inverse are sufficient to convert any two special spines of the same compact three-dimensional manifold to one another, see [12]; this fact is crucial for the construction of the Turaev–Viro invariants [28].

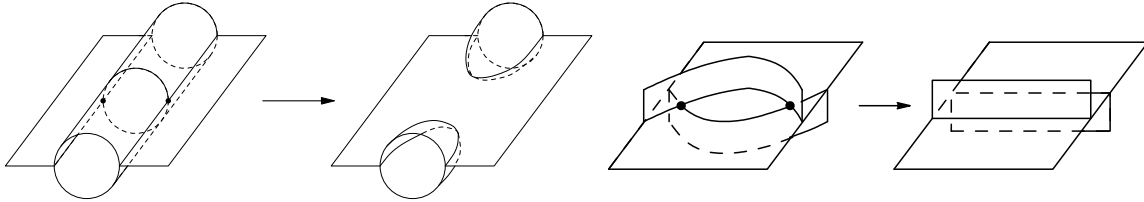


FIGURE 9. Two presentations of another simplification move

Figure 9 represents another simplification move, which is applicable to spines containing a component with short boundary of length 2; clearly, the neighborhood of this component looks like the left hand side of Fig. 9. This simplification move yields a simple, but not necessarily special, spine of the same manifold (provided that the move had been applied to a simple spine).

To construct a spine of $M(\mathcal{A})$, consider a fiber $T^2 \times \{0\}$ and choose a θ -curve L_0 in it; by doing so, we also fix a θ -curve L_1 in $T^2 \times \{1\}$; note that $W(L_1) = \mathcal{A}W(L_0)$. This choice is equivalent to the choice of some basis in the lattice $H_1(T^2, \mathbb{Z})$; by A

denote the matrix of \mathcal{A} in this basis. Construct a continuous family L_t transforming L_0 into L_1 by isotopy and $c(\mathcal{A})$ flips. Set $P_0 = \bigcup_{t \in [0,1]} L_t$; we assume that each L_t

is embedded in $T^2 \times \{t\}$. Note that P_0 is a simple polyhedron, which is a spine of some punctured torus bundle. Two-dimensional components of P_0 come from edges of L_t as t varies; similarly, one-dimensional components of P_0 come from vertices of L_t . The $c(\mathcal{A})$ flips correspond to the vertices of P_0 , see Fig. 10.

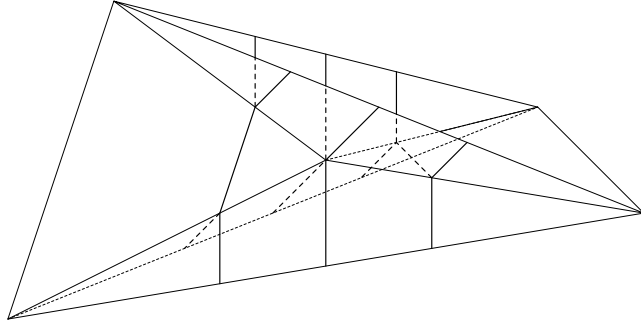


FIGURE 10. Vertices correspond to flips

To minimize the number of vertices, it is natural to choose a basis in $H_1(T^2, \mathbb{Z})$ so that the operator A is minimal for \mathcal{A} . If the operator A is not minimal, Theorem 7 guarantees that the last flip of the first round along the base circle of the fibering and the first flip of the second round are mutually inverse. This means that the second simplification move (see Fig. 9) is applicable. Apply it until it is no longer possible. This process is nothing but the construction of a minimal hexagon for \mathcal{A} by the algorithm described below the proof of Theorem 7. In the following, we suppose that A is a minimal matrix.

Examples. 1. The only periodic operators \mathcal{A} with $c(\mathcal{A}) > 0$ have the minimal matrices $\begin{pmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{pmatrix}$. This is a very interesting case. The polyhedron P_0 constructed above has one vertex. Consider the two-sheeted covering of the base S^1 of the fibering. It induces the two-sheeted covering of the total space by the manifold $M(\mathcal{A}^2) = M(-I)$. The preimage of P_0 under the covering is a polyhedron in $M(-I)$ with two vertices, which can be cancelled by the second simplification move in two different ways. This is the only (up to a sign and conjugacy) operator such that $c(\mathcal{A}^k) < |k|c(\mathcal{A})$ for $k \in \mathbb{Z}$, $|k| \geq 2$, see Theorem 8; the other periodic operators are of complexity 0.

2. If $c(\mathcal{A}) = 0$, there are no flips at all. In this case P_0 contains no vertices and consists of three orientable annuli and three edges if $A = I$, of three nonorientable annuli and one edge if $A = -I$, of one nonorientable annulus and one edge if A is equal to the standard hexagon rotation matrix $R_{\pi/3} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ or its inverse (note that $R_{\pi/3}^6 = I$), and of one orientable annulus and two edges if A equals $R_{\pi/3}^2$ or $R_{\pi/3}^4$. This exhausts the case $c(\mathcal{A}) = 0$.

The polyhedron P_0 is not a spine of $M(\mathcal{A})$, because the fibered space $M(\mathcal{A})$ admits a section that does not intersect P_0 . This section represents a nontrivial element of the group $\pi_1(M(\mathcal{A}))$, while the complement to a spine of a closed

manifold is a cell and hence cannot contain nontrivial loops. Let us put $P_1 = P_0 \cup (T^2 \times \{0\})$.

Lemma 2. P_1 is a spine of $M(\mathcal{A})$.

Proof. It is sufficient to show that $M(\mathcal{A}) \setminus P_1$ is a 3-dimensional cell. We have $M(\mathcal{A}) \setminus P_1 = T^2 \times (0, 1) \setminus P_0 = T^2 \times (0, 1) \setminus \bigcup_{t \in (0, 1)} L_t = \bigcup_{t \in (0, 1)} (T^2 \times \{t\} \setminus L_t)$, and

Lemma follows. \square

Note that P_1 is not a simple polyhedron. Indeed, its part $T^2 \times \{0\}$ contains a singular subset L_0 , which is more complicated than a triple line: three edges of L_0 yield three lines of transversal intersection of two surfaces, and any of two vertices of L_0 gives rise to a transversal intersection of a triple line with one extra surface.

Let us modify the previous construction by gluing $T^2 \times \{0\}$ with $T^2 \times \{1\}$ along a homeomorphism $\mathcal{A} + \delta$, where δ is a small shift of the torus in a direction transversal to the edges of L_0 , see Fig. 11. Put $P_2 = \bigcup_{t \in [0, 1]} L_t \cup (T^2 \times \{0\})$. Again, P_2 is a spine of $M(\mathcal{A})$.

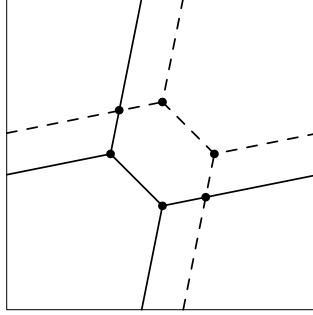


FIGURE 11. θ -curve L_0 and its δ -shift L_1 (dashed line) in the fiber $T^2 \times \{0\}$ of $M(\mathcal{A})$

Lemma 3. P_2 is a special spine of $M(\mathcal{A})$ with $c(\mathcal{A}) + 6$ vertices.

Proof. It is clear from the construction that P_2 is a simple polyhedron. Its triple lines are the “trajectories” (as t varies) of the vertices of L_t and the ten segments of L_0 and L_1 shown on Fig. 11, where the torus is represented by a square with the opposite sides to be identified. There are $c(\mathcal{A})$ vertices of P_2 that correspond to $c(\mathcal{A})$ flips between L_0 at $t = 0$ and L_1 at $t = 1$, and six other vertices that are drawn on Fig. 11. Two of them arise from $T^2 \times \{0\}$ and L_t , $0 \leq t < \varepsilon$, and their neighborhoods in P_2 look like Fig. 1 d. Two other vertices on Fig. 11 arise from $T^2 \times \{0\} = T^2 \times \{1\}$ and L_t , $1 - \varepsilon < t \leq 1$; their neighborhoods look like the horizontal mirror reflection of Fig. 1 d. The last two vertices on Fig. 11 correspond to two intersection points of L_0 and L_1 , and their neighborhoods look like Fig. 1 c. Thus P_2 is a simple spine of $M(\mathcal{A})$ with $c(\mathcal{A}) + 6$ vertices.

It remains to prove that SP_2 contains no closed triple lines and all connected components of $P_2 \setminus SP_2$ are 2-dimensional cells, cf. Definition 2. First group of triple lines of SP_2 is formed by ten arcs in $T^2 \times \{0\}$ shown on Fig. 11. Obviously, they are not closed. The rest $2c(\mathcal{A}) + 2$ triple lines are swept by the vertices of the θ -curves $L_t \subset T^2 \times \{t\}$, $0 < t < 1$. They end at vertices of P_2 , too, and thus are not closed.

Connected components of $P_2 \setminus SP_2$ also belong to two groups. Four of them, two hexagonal and two quadrilateral, lie in the fiber $T^2 \times \{0\}$, see Fig. 11. They are cells. Any other connected component of $P_2 \setminus SP_2$ intersects any fiber $T^2 \times \{t\}$, $a < t < b$ (where a is equal to either 0 or one of the flip moments, and b is either one of the flip moments or is equal to 1), along one edge of L_t , and does not intersect other fibers; this implies that this component is a cell. We have proved that the polyhedron P_2 is special. \square

Corollary. $c(M(\mathcal{A})) \leq c(\mathcal{A}) + 6$. \square

Example. Three-dimensional torus can be represented as $M(I)$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Since $c(I) = 0$, the construction above gives a special spine of T^3 with six vertices. The manifold $T^3 = M(I)$ is contained in Table 7 of the preprint [14] under the name 6_{71} . It is shown in [14] that all manifolds of complexity at most 5 are different from T^3 . So we have $c(T^3) = 6$. The spine that we constructed here does not differ from the spine 6_{71} from [14, §5.2], while our way of presenting spines differs significantly from the one used in [14].

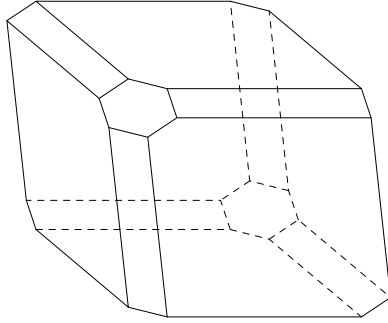


FIGURE 12. The complement $T^3 \setminus P_2$ of the minimal spine P_1 of T^3

The torus T^3 can be obtained from the cube by gluing its opposite faces. This yields a natural cell decomposition of T^3 with one vertex, three edges, three “square” 2-dimensional cells and one 3-dimensional cell. The 2-dimensional skeleton $\text{sk}_2(T^3)$ has singular points more complicated than triple lines and vertices of simple polyhedra. However, the minimal spine of T^3 can be obtained as a small perturbation of $\text{sk}_2(T^3)$.

Let the θ -curves L_t , $t \in [0, 1]$, be very close to the bouquet of a parallel and a meridian of $T^2 \times \{t\}$, and let the shift δ involved in the construction of P_2 be very small. Then the 3-dimensional cell $T^3 \setminus P_2$ is very close to the 3-dimensional cube. Figure 12 represents this cell. If we identify opposite faces of this polyhedron by parallel transports (or, equivalently, tessellate \mathbb{R}^3 into parallel copies of this polyhedron and consider a quotient over the appropriate lattice \mathbb{Z}^3), we get the torus T^3 ; the image of the boundary of the polyhedron under this gluing is the minimal spine of T^3 close to $\text{sk}_2(T^3)$.

The same construction gives special spines with six vertices for the manifolds $M(-I) = 6_{70}$,

$$M\left(\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\right) = M\left(\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\right) = 6_{67},$$

and

$$M\left(\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\right) = M\left(\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\right) = 6_{65}.$$

The spines constructed in this way are minimal spines of these manifolds, because all of them are of complexity 6; in fact, all manifolds of complexity up to 5 are quotient spaces of the sphere S^3 , see [14].

However, in all other cases (that is, if $c(\mathcal{A}) > 0$) the spines with $c(\mathcal{A}) + 6$ vertices are not minimal spines of the manifolds $M(\mathcal{A})$. For example, the spaces

$$6_{66} = M\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right), \quad 6_{68} = M\left(\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}\right), \quad \text{and} \quad 6_{69} = M\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right)$$

are manifolds of complexity 6, while

$$c\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = c\left(\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}\right) = c\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right) = 1,$$

and our construction gives their spines with 7 vertices.

This happens because some of the spines with $c(\mathcal{A}) + 6$ vertices constructed above are not pseudominimal whenever $c(\mathcal{A}) > 0$. Namely, they have a triangular component, and the first simplification move (see Fig. 8) can be applied.

Let us return to Fig. 11. Assume that the first flip in the sequence taking L_0 to L_1 involves the short edge of L_0 , that is, the edge that does not intersect dashed lines on Fig. 11. This condition can be satisfied by an appropriate choice of the shift δ involved in the construction of P_2 . Then the 2-dimensional cell of P_2 adjacent to this edge and not contained in $T^2 \times \{0\}$ is a triangle, and we can apply the first simplification move, which gives a spine of $M(\mathcal{A})$ with a smaller number of vertices. This spine can be described in other words as follows. Let L' be the θ -curve obtained after the first of $c(\mathcal{A})$ flips converting L_0 into L_1 . Glue the square from Fig. 13 into the torus $T^2 \times \{0\} \in M(\mathcal{A})$ and embed L_t in $T^2 \times \{t\}$ for all $t \in (0, 1)$, where the family L_t contains $c(\mathcal{A}) - 1$ flips and connects L' with L_1 . Note that the first of $c(\mathcal{A}) - 1$ flips converting L' into L_1 is performed along a long edge of L' , because a flip along the short edge would annihilate with the flip converting L_0 to L' . The new spine P_3 has $6 + c(\mathcal{A}) - 1 = c(\mathcal{A}) + 5$ vertices. So we have $c(M(\mathcal{A})) \leq c(\mathcal{A}) + 5$ whenever $c(\mathcal{A}) > 0$.

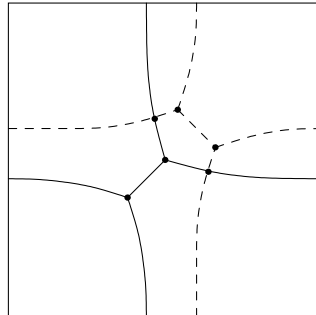


FIGURE 13. θ -curves L' and L_1 (dashed) in $T^2 \times \{0\}$

The proof that the spine P_3 is special repeats the proof of Lemma 3.

Theorem 10.

- 1) $c(M(\mathcal{A})) \leq \max(6, c(\mathcal{A}) + 5)$.
- 2) *The spine P_3 constructed above is pseudominimal.*

Compare the first statement with Corollary of Lemma 3.

Proof. If $c(\mathcal{A}) = 0$, then $c(M(\mathcal{A})) = 6$. So we may suppose that $c(\mathcal{A}) > 0$. Since P_3 is an almost simple (and even special) spine of $M(\mathcal{A})$ with $c(\mathcal{A}) + 5$ vertices, the first statement is obvious. The argument similar to the proof of Lemma 3 shows that two-dimensional cells of P_3 have no counterpasses. So we only have to show that P_3 has no components with short boundaries. The four cells contained in $T^2 \times \{0\}$ are pentagons, see Fig. 13. The cells that have no boundary edges in $T^2 \times \{0\}$ have even numbers of edges, namely, $2k - 2$, where k is the number of flips from the vertex where the cell appears to the vertex where the cell disappears (including both the first flip and the last one). The matrix A involved in the construction of P_3 is a minimal matrix of \mathcal{A} . This implies that any two consecutive flips in the sequence involved in the construction are not inverse to one another, that is, $k > 2$ for any cell considered above.

It remains to consider at most 6 two-dimensional cells that have an edge in $T^2 \times \{0\} = T^2 \times \{1\}$ (if A is, up to a sign, a power of a Jordan block, there are only 5 cells of this type; otherwise, no 2-cell touches both $T^2 \times \{0\}$ along an edge of L' and $T^2 \times \{1\}$ along an edge of L_1 , so three edges of L' and three edges of L_1 belong to six different cells of $P_3 \setminus T^2 \times \{0\}$). Two cells of $P_3 \setminus T^2 \times \{0\}$ are adjacent to the long edges of L' (the edges that intersect dashed lines on Fig. 13). Each of these cells has at least 4 boundary edges: two segments of a long edge of L' and two edges (transversal to fibers) that arise from the vertices of L' . The same argument works for two cells of $P_3 \setminus T^2 \times \{0\}$ adjacent to the long edges of L_1 . Consider the cell of $P_3 \setminus T^2 \times \{0\}$ adjacent to the short edge of L' . Of course, it has at least 3 edges: the short edge of L' and two edges that are trajectories of the vertices of L' as t varies. It has at least one more edge: otherwise, the first flip in the sequence of flips converting L' to L_1 is performed along the short edge of L' , which is impossible by the construction of P_3 , see above. For the same reason, the last flip (which results in L_1) cannot be performed along the short edge of L_1 : by the minimality of the matrix A , it cannot be cancelled with the flip connecting L_0 with L' , see Theorem 7. This means that the 2-dimensional cell of $P_3 \setminus T^2 \times \{1\}$ adjacent to the short edge of L_1 also has more than 3 edges, and P_3 contains no components with short boundaries. The Theorem is proved. \square

Conjecture 2. *The pseudominimal spines of the manifolds $M(\mathcal{A})$ constructed above are in fact their minimal spines, and the upper bound for complexity given in Theorem 10 is in fact its exact value: $c(M(\mathcal{A})) = \max(6, c(\mathcal{A}) + 5)$ for any monodromy operator $\mathcal{A} \in \text{SL}(2, \mathbb{Z})$. In other words, any singular triangulation of $M(\mathcal{A})$ involves at least $c(\mathcal{A}) + 5$ tetrahedra if $c(\mathcal{A}) > 0$ and 6 tetrahedra if $c(\mathcal{A}) = 0$.*

2.4. Digression: spines of lens spaces.

Pseudominimal special spines of the lens spaces $L_{p,q}$, $p > 3$, with exactly $E(p, q) - 3$ vertices were constructed in [14]. In that paper, spines are presented by drawing the neighborhood of the singular graph of a spine. This allows to draw spines on the plane; however, it remains unclear how the spines are embedded into corresponding manifolds.

In this section, we construct pseudominimal special spines of $L_{p,q}$, $p > 3$, with $E(p, q) - 3$ vertices, making use of the techniques developed in §2. We omit some details and proofs.

Consider two solid tori. The meridians of their boundary tori are well defined, while the parallels are defined modulo meridians only. Let μ_0, μ_1 be the meridians of the tori and σ_0, σ_1 be their parallels such that the pair of the oriented cycles (σ_0, μ_0) defines the positive orientation of the boundary of the first torus and the pair (σ_1, μ_1) defines the negative orientation of the boundary of the second torus. There is a unique pair of positive integer numbers (r, s) such that $r < p$, $s < q$, and $qs - pr = 1$. Put $A = \begin{pmatrix} s & p \\ r & q \end{pmatrix}$ and attach the solid tori to one another so that the induced homomorphism of the one-dimensional homology groups of their boundary tori has the matrix A (in the bases (σ_0, μ_0) and (σ_1, μ_1)). We get a closed orientable 3-manifold that is nothing but $L_{p,q}$.

Note that $A \in \text{SL}(2, \mathbb{Z})$, $c(A) = E(p, q)$, and the parallels σ_0 and σ_1 represent nontrivial elements of $\pi_1(L_{p,q}) = \mathbb{Z}_p$. This implies that any spine of $L_{p,q}$ intersects these loops. Let us shift σ_0 in the interior of the first solid torus and consider the tubular neighborhood U_0 of the shifted curve. Obviously, U_0 is a solid torus. Similarly, construct U_1 as a tubular neighborhood of σ_1 shifted inside of the interior of the second torus. We may assume U_0 and U_1 to be disjoint. Then $L_{p,q} = U_0 \cup (T^2 \times [0, 1]) \cup U_1$. Let L_i , $i = 0, 1$, be standard (with respect to the bases (σ_i, μ_i)) θ -curves in the tori $T_i^2 = \partial U_i$; they are defined up to isotopy. Following the construction of §2.3, consider a continuous family $L_t \subset T^2 \times \{t\}$ connecting L_0 to L_1 with $c(A)$ flips. Put $P_0 = \bigcup_{t \in [0, 1]} L_t$. Let D_i , $i = 0, 1$, be meridional disks of

the U_i intersecting L_i transversally at one point. Put $P_1 = D_0 \cup T_0^2 \cup P_0 \cup T_1^2 \cup D_1$.

Lemma 4. *The polyhedron P_1 is a special spine of $L_{p,q}$ with three punctures. It has $E(p, q) + 6$ vertices.*

Proof. The complement $L_{p,q} \setminus P_1$ consists of three cells $U_0 \setminus D_0$, $(T^2 \times [0, 1]) \setminus P_0$, and $U_1 \setminus D_1$. There are $c(A) = E(p, q)$ vertices in the interior part of P_0 . Further, there are 3 vertices on T_0^2 , which correspond to two vertices of L_0 and the intersection point of L_0 and ∂D_0 . Similarly, there are 3 vertices of P_1 on T_1^2 . It remains to show that P_1 is a special polyhedron. This can be proven by analogy with Lemma 3. \square

Below we show that one can decrease the number of vertices “inside of P_0 ” by one and the number of vertices “near each U_i ” by four. This gives a spine with $E(p, q) + 6 - 1 - 4 - 4 = E(p, q) - 3$ vertices.

Recall that the parallels σ_i are defined only modulo meridians μ_i . Thus, the θ -curves L_i are defined only up to powers of the Dehn twists along the meridians, that is, up to transformations $\sigma_i \mapsto \sigma_i + n_i \mu_i$, $n_i \in \mathbb{Z}$. By varying n_0 and n_1 , one can decrease the distance in Γ between $B^{n_0}W_0$ and $C^{n_1}AW_0$ and thus decrease the number of the vertices inside of P_0 ; here W_0 is the standard hexagon, $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is the matrix representing the Dehn twist along μ_0 , and $C = ABA^{-1}$ is the matrix corresponding to the Dehn twist along μ_1 .

Lemma 5. $\min_{n_0, n_1 \in \mathbb{Z}} d(B^{n_0}W_0, C^{n_1}AW_0) = E(p, q) - 1$.

Proof. Note that the graph Γ contains edges $B^u W_0 B^{u+1} W_0$ and $C^v A W_0 C^{v+1} A W_0$ for all $u, v \in \mathbb{Z}$. Since Γ is a tree, there are $m_0, m_1 \in \mathbb{Z}$ such that the path

from $B^{n_0}W_0$ to $C^{n_1}AW_0$ for all $n_0, n_1 \in \mathbb{Z}$ consists of the following three legs: $B^{n_0}W_0 B^{m_0}W_0$, $B^{m_0}W_0 C^{m_1}AW_0$, and $C^{m_1}AW_0 C^{n_1}AW_0$. Now it is obvious that $\min_{n_0, n_1 \in \mathbb{Z}} d(B^{n_0}W_0, C^{n_1}AW_0) = d(B^{m_0}W_0, C^{m_1}AW_0)$. By considering the three legs of the path from W_0 to AW_0 , one can see that $m_0 = 1$ (because $p > q > 0$), $m_1 = 0$ (because both positive and negative Dehn twists along μ_1 do not affect the leading vertex (p, q) of AW_0 and thus increase the distance to W_0), and the length of the middle leg of this path is $d(BW_0, AW_0) = d(W_0, AW_0) - |m_0| - |m_1| = E(p, q) - 1$. Also see Theorem 17 in §4.3. \square

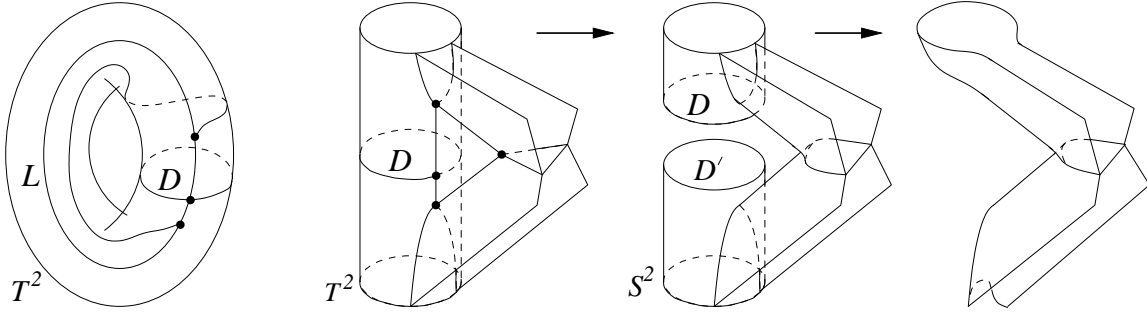


FIGURE 14. Simplification of P_0 near T_i^2

By virtue of Lemma 5, we can decrease by one the number of the vertices inside of P_0 by another choice of a θ -curve L_0 . Now we have a special spine of $L_{p,q}$ with three punctures having $E(p, q) + 5$ vertices. The disks D_0 and D_1 are components with short boundaries. By τ_i denote the edge of L_i that intersects ∂D_i . Note that two other edges of L_i form the meridian of T_i , the first flip in the sequence connecting L_0 with L_1 is performed along τ_0 while the last flip in this sequence is performed along τ_1 (flips along other edges are equivalent to meridional Dehn twists and thus do not lead out of the mainstreams $\gamma(B) = \{B^{n_0}W_0 \mid n_0 \in \mathbb{Z}\}$ and $\gamma(C) = \{C^{n_1}AW_0 \mid n_1 \in \mathbb{Z}\}$, while the path between L_0 and L_1 is the shortest path that connects these mainstreams). We can apply the following simplification move in the neighborhood of D_i . First, add a parallel copy D'_i of D_i . Second, delete the lateral surface of the cylinder bounded by D_i , D'_i , and a thin strip of T_i^2 . Finally, delete the cell of P_1 adjacent to τ_i ; this cell is triangular, because τ_i is the edge involved in the flip in P_0 closest to T_i^2 , see Fig. 14. So, the first step adds one vertex on each T_i^2 , the second step kills two vertices on each T_i^2 , and the last step kills three vertices near each of T_i^2 . By P_1 denote the polyhedron obtained by the construction above. Obviously, it has $E(p, q) - 3$ vertices. Further, one can see that two remaining edges of L_i (which differ from τ_i) form a closed triple line S_i^1 and the complement $L_{p,q} \setminus P_1$ still consists of three cells, two of which are bounded by the spheres S_i^2 obtained from the torus T_i^2 and their meridional disks D_i , D'_i by deleting the thin strip bounded by ∂D_i and $\partial D'_i$ from T_i^2 . The circles S_i^1 divide the spheres S_i^2 into two disks each; one of the disks contains D_i , the other contains D'_i . Delete from P_1 the disks of the S_i^2 that contain D'_i . This yields a polyhedron P with $E(p, q) - 3$ vertices such that $L_{p,q} \setminus P$ is a cell.

Theorem 11. *The polyhedron P is a pseudominimal special spine of $L_{p,q}$ with $E(p, q) - 3$ vertices. It coincides with the spine of $L_{p,q}$ presented in [14]. \square*

It was shown in [14] that the spines constructed in that paper are pseudominimal. So it is sufficient to prove only the second statement; we leave it to the reader.

§3. LOWER BOUND FOR C^1 -SMOOTH SPINES TRANSVERSAL TO FIBERS

In this section, we prove the following statement.

Theorem 12. *Let \mathcal{A} be a non-periodic operator. Then for any C^1 -smooth spine P of $M(\mathcal{A})$ transversal to the fibers, we have $c(P) \geq \frac{1}{5}c(\mathcal{A}) + 2$, where $c(P)$ stands for the number of the vertices of P .*

Recall that $M(\mathcal{A})$ is the total space of the fibering $p: M^3 \xrightarrow{T^2} S^1$ with monodromy operator \mathcal{A} .

3.1. Morse transformations in simple polyhedra.

Projection $p: M(\mathcal{A}) \rightarrow S^1$ defines an S^1 -valued function on $M(\mathcal{A})$. Let us explore its restriction $p|_P$ to a simple spine P of $M(\mathcal{A})$. The polyhedron P is a stratified space; its strata are the vertices, the edges, and the two-dimensional components of P . We may assume that $p|_{\bar{\sigma}}$ has an everywhere continuous derivative for any edge or 2-component σ : a C^0 -small deformation of the embedding mapping $i: P \hookrightarrow M^3$ is sufficient. Since the space of Morse functions on a manifold is C^1 -dense in the set of all C^1 -smooth functions on it [16], we may assume that $p|_{\bar{\sigma}}$ is a Morse function for any edge or 2-component σ of P . In this case p is called a *Morse function* on P .

Consider a fiber $F_t = p^{-1}(t)$ of the fibering $p: M(\mathcal{A}) \rightarrow S^1$, where t is a local coordinate on S^1 . If t is a nonsingular value of p (that is, F_t contains neither vertices of P nor critical points of the restrictions of p to the edges and 2-components of P), then F_t is transversal to all strata of P and the intersection $K_t = F_t \cap P$ is a trivalent graph, possibly disconnected. We will explore how K_t changes as t varies. For much more comprehensive exposition of Morse transformations in stratified spaces, see [9].

Without loss of generality, it can be assumed that any singular fiber (that is, a singular level of p) contains only one singular point of $p|_P$ (including vertices), and there are only finite number of singular fibers. Also, we may assume that the restrictions $p|_{\bar{\sigma}}$ to the closures of triple lines and two-dimensional components of P have no boundary critical points. Let two nonsingular fibers F_- and F_+ be the preimages of two close points $t_-, t_+ \in S^1$. If the interval (t_-, t_+) contains no singular values of p , then the graphs K_- and K_+ are isotopic, that is, there is a continuous family of embeddings $i_t: K \hookrightarrow F_t$, $t \in [t_-, t_+]$, of the same graph K to the tori F_t such that $K_- = i_{t_-}(K)$ and $K_+ = i_{t_+}(K)$.

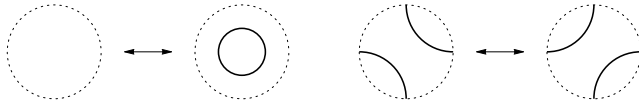


FIGURE 15. Minimax (left) and saddle (right) Morse transformations

Now suppose that there is exactly one singular fiber F_0 between F_- and F_+ ; by t_0 denote the corresponding critical value. Thus F_0 contains either a singular point of the restriction of p to a two-dimensional component of P or a singular

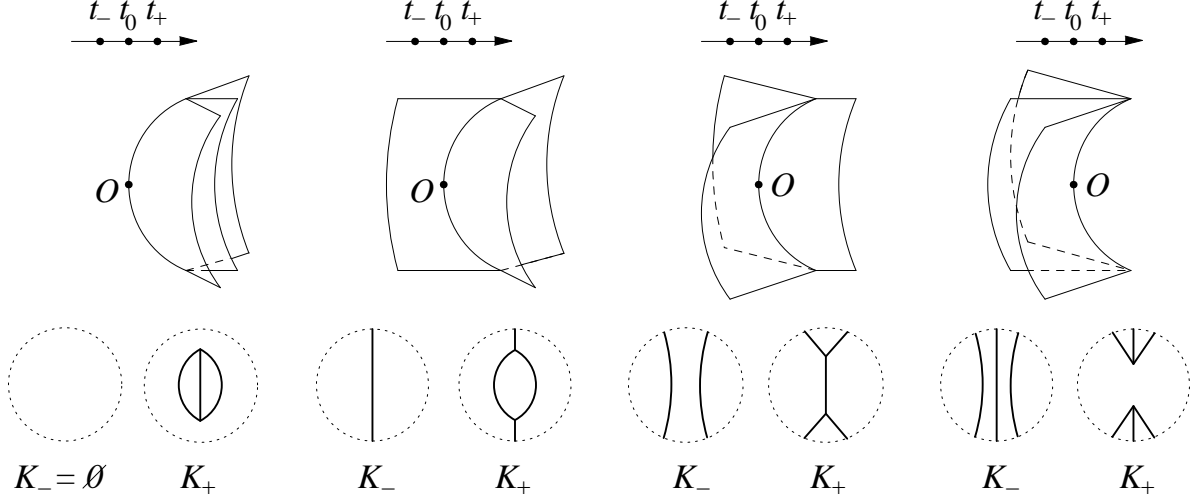


FIGURE 16. Transformation of K induced by a minimum point on a triple line

point of the restriction of p to a triple line of P , or a vertex of P . In the first case, the difference between K_- and K_+ is nothing but the Morse transformation of the level set $Q = t - t_0$ of a real quadratic form $Q(x, y) = \pm x^2 \pm y^2$, see Fig. 15.

To explore the second case, suppose that the critical point of the restriction of p to a triple line of P is a minimum point of p ; denote this point by O . By v_i , $i = 1, 2, 3$, denote the unit vector tangent to the i th two-dimensional component σ_i at O and orthogonal to the triple line. Since O is not a critical point of $p|_{\sigma_i}$, we have $p_*v_i \neq 0$. Here the σ_i are the components of $(P \setminus SP) \cap U(O)$ of the two-dimensional strata of P in the neighborhood $U(O)$ of O ; it does not matter whether the σ_i actually belong to different components of $P \setminus SP$. Let us say that the component σ_i is *ascending* at O if the vector $p_*v_i \in T_{t_0}S^1$ induces a positive orientation of S^1 and *descending* if this orientation is negative. There may be 0, 1, 2 or 3 descending components. Their number defines the transformation between K_- and K_+ up to isotopy. The graphs K_- and K_+ differ only inside of the dotted circle, see Fig. 16. If O is a maximum point of p restricted to a triple line, one has to revert all arrows and signs in the lower indices on Fig. 16.

The last case (where the fiber F_0 contains a vertex $V \in P$) is more complicated. We will return to it later.

Remark. The double of a disk is S^2 . Draw the fragment $K_- \cap U(O)$ of the graph K_- (that is, the picture inside of the left dotted circle, see Figs. 15, 16) on the lower hemisphere of S^2 and the fragment $K_+ \cap U(O)$ of the graph K_+ (that is, the picture inside of the right dotted circle) on the upper hemisphere. In all four cases, we get an embedding of the link of O in P to the sphere with the equator drawn on it by a dotted line. The number of intersection points of the equator and the link of O is twice the number of descending components at O .

This is not a mere coincidence but a general recipe for describing the difference between K_- and K_+ . Suppose that there is exactly one singular point O between F_- and F_+ . Consider an ε -neighborhood $U(O)$ of O , where $\varepsilon < t_0 - t_-$ and $\varepsilon < t_+ - t_0$. The intersection $P \cap U(O)$ is the cone over the graph $\text{lk } O$, which is the circle with none, two or three radii, see the definition of a simple polyhedron (Definition 1 in §1). By D_- , D_0 , and D_+ denote the intersections of $U(O)$ with

F_- , F_0 , and F_+ , respectively. Obviously, $K_+ \setminus D_+$ is isotopic to $K_- \setminus D_-$. Thus, the difference between K_+ and K_- is “hidden inside of $U(O)$ ”. The singular fiber F_0 cuts the sphere $S^2 = \partial U(O)$ into two hemispheres S_-^2 and S_+^2 . Put $K'_- = (K_0 \setminus U(O)) \cup (P \cap S_-^2)$ and $K'_+ = (K_0 \setminus U(O)) \cup (P \cap S_+^2)$. These graphs are embedded into tori $(F_0 \setminus U(O)) \cap S_-^2$ and $(F_0 \setminus U(O)) \cap S_+^2$.

Lemma 6. *The graph K_- , respectively, K_+ , is isotopic to the graph K'_- , respectively, K'_+ .*

Proof. Let us define a surface G_s as the union of $F_{t_0+s(t_0-t_-)} \setminus U(O)$ and $\{x \in S_-^2 \mid p(x) < t_0 + s(t_0 - t_-)\}$, where $-1 \leq s \leq 0$, and put $K'_s = P \cap G_s$. In $U(O) \setminus O$, the edges and 2-components of P are transversal to all fibers and to the spheres centered at O . So all G_s , $-1 \leq s \leq 0$, are transversal to P , and the family K'_s provides an isotopy between $K'_{-1} \subset G_{-1}$ and $K'_0 \subset G_0$. It remains to note that $G_{-1} = F_-$, $K'_{-1} = K_-$, and $K'_0 = K'_-$. An isotopy between K_+ and K'_+ can be constructed in a similar way. \square

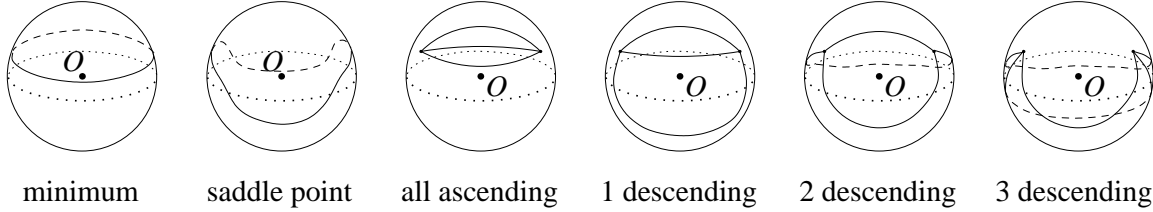


FIGURE 17. Mutual position of $\text{lk}_{F_0}(O)$ (dotted line) and $\text{lk}_P(O)$ in S^2

So to describe a “surgery” in the graph K near a singular point O , it is sufficient to draw the links of O in F_0 and in P on the sphere $S^2 = \partial U(O)$ and cut S^2 into two disks S_-^2 and S_+^2 along the former link (the equator). The mutual position of $\text{lk}_{F_0} O$ and $\text{lk}_P O$ may be considered as a real analogue for the Milnor fiber of an isolated singularity on a complex hypersurface, see [17]. Figure 17 represents the mutual position of these links on S^2 for minimax and saddle points on a two-dimensional component of P and for the four cases of Fig. 16, where O is a minimax point of the restriction of p to an edge of P . Note that there are, up to isotopy and symmetries, exactly four different embeddings of a circle with a diameter into a sphere with an equator, provided that any edge of the graph intersects the equator at most twice and both vertices lie in the same hemisphere.

The same argument works if a singular point is a vertex V of a simple polyhedron P . In this case, $\text{lk}_P(V)$ is a circle with three radii; denote this graph by Δ . Any edge of Δ is the link of V in one of the six two-dimensional components of $(P \setminus SP) \cap U(V)$, and we assumed that V is not a critical point of the restriction of p to the closures of these components. This implies that any edge of Δ intersects the equator at most twice, because it is close to an arc of a great circle obtained as the intersection of $S^2 = \partial U(V)$ and the tangent plane to a component σ at V . Since the differential $dp|_{\bar{e}}$ does not vanish at V for any edge e of P , the vertices of Δ lie in either S_+^2 or S_-^2 but not at the equator $\text{lk}_{F_0}(V)$.

Without loss of generality, we can assume that $P \cap U(V)$ consists of plane pieces. Now it follows that any edge of Δ connecting vertices in different hemispheres intersects the equator exactly once. An edge with both endpoints in the same

hemisphere can have either none or two intersection points with the equator; in the latter case, these two points are opposite. An edge is said to be *long* if it intersects the equator twice. We have to consider three cases.

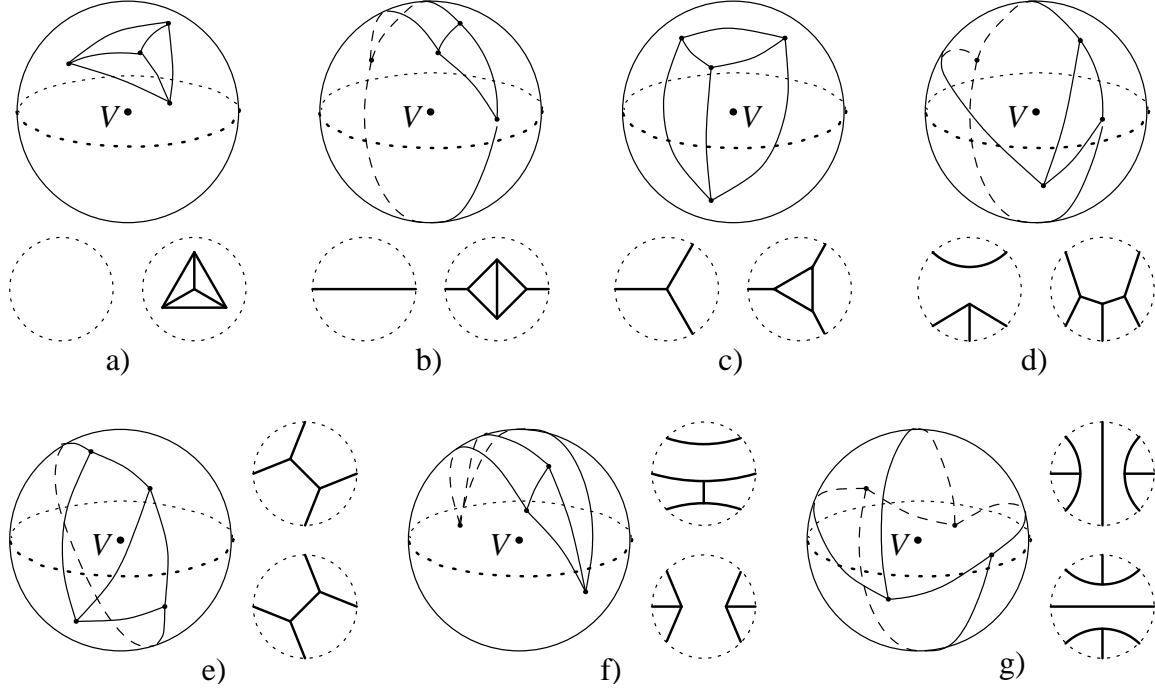


FIGURE 18. Transformation of K near a vertex V

Case 1. All four vertices of Δ lie in one hemisphere. Then any long edge contains a diameter of the other hemisphere. Since any two diameters intersect one another, there is at most one long edge. Thus in Case 1 there are, up to isotopy, only two possibilities to draw Δ on a sphere with the equator, see Fig. 18 a,b.

Figure 18 also shows the difference between K_- and K_+ .

Case 2. Three vertices of Δ lie in one hemisphere and the last one in the other. Again, there is at most one long edge and thus only two nonisotopic embeddings of Δ , see Fig. 18 c,d.

Case 3. There are two vertices in upper hemisphere and two other in the lower one. The maximal number of long edges is two: at most one in each hemisphere. If there are no long edges, we have the situation of Fig. 18 e; if there is one long edge, the picture is like Fig. 18 f; finally, if there are two long edges, we obtain the situation of Fig. 18 g.

Note that all results of Section 3.1 apply to the level sets of Morse functions on arbitrary simple polyhedra, not only to simple spines of $M(\mathcal{A})$.

3.2. θ -curves in the fibers.

Recall that a spine P of a closed 3-manifold M intersects any loop that is nontrivial in $\pi_1(M)$.

Lemma 7. *For any t , the graph $K_t = P \cap F_t$ intersects any loop representing a nonzero element of $\pi_1(F_t) = \mathbb{Z}^2$.*

Proof. Consider the exact sequence

$$\dots \rightarrow \pi_2(S^1) \rightarrow \pi_1(T^2) \xrightarrow{i_*} \pi_1(M^3) \rightarrow \dots$$

of the fibering $p: M^3 \xrightarrow{T^2} S^1$. Since $\pi_2(S^1) = 0$, it follows that i_* is a monomorphism, that is, any nontrivial loop in T^2 is nontrivial in M^3 , too. Hence any spine of M intersects this loop, and the Lemma follows. \square

We assumed that p is an S^1 -valued Morse function on P . In this case, all critical points of p are isolated, and all but finite number of values of p are regular. Let F be a fiber corresponding to a regular value of p . Then the intersection $F \cap P$ is a trivalent graph $K \subset F$.

Lemma 8. *Suppose that a trivalent graph $K \subset T^2$ intersects any nontrivial loop in T^2 . Then K contains a subgraph that is a θ -curve.*

Proof. Let K_1, \dots, K_n be connected components of K . If a component K_i contains no cycles nontrivial in T^2 , then there is a disk $D_i^2 \subset T^2$ such that $K \cap D_i^2 = K_i$. If there are several components K_i without nontrivial cycles, then there exists a disjoint union of disks U such that $K \cap U$ coincides with the set of all connected components of K containing no nontrivial cycles. Put $K' = K \setminus U$. All connected components of K' (if any) contain nontrivial cycles.

Any cycle in $\pi_1(T^2)$ is homotopic to a cycle contained in $T^2 \setminus U$. This means that any nontrivial cycle intersects K' , and thus $K' \neq \emptyset$. Since any component K'_i of K' contains nontrivial cycles, we can choose some cycles $\gamma_i \subset K'_i$ represented by simple closed curves. If there are several connected components of K' , the cycles γ_i do not intersect one another and thus are homotopic. In this case, among connected components of $T^2 \setminus K'_1$ there is an annulus containing other components of K' . It contains a cycle γ (homotopic to all the γ_i) going along a boundary circle of the annulus. The cycle γ is nontrivial and does not intersect K . Since this is impossible, the graph K' is connected.

Let U_1, \dots, U_k be the connected components of $T^2 \setminus K'$. Every U_i is an orientable surface with boundary. It contains no closed curves nontrivial in U_i . Otherwise, such a curve would be either trivial or nontrivial in T^2 . In the former case, it would split the torus into two disjoint parts that contain different connected components of K' , which is impossible since the graph K' is connected. The latter case is impossible, too, because any cycle nontrivial in T^2 intersects K' . A surface with boundary containing no nontrivial cycles is a disk. Thus $T^2 \setminus K'$ is a collection of s 2-dimensional cells, and K' defines a cell decomposition of T^2 .

Let v , respectively, e , be the number of vertices, respectively, edges of K' . Note that $e = \frac{3}{2}v$, because K' is a trivalent graph. Also, we have $v - e + s = \chi(T^2) = 0$, which implies that $v = 2s$ and $e = 3s$. If the number s is greater than 1, it can be decreased by deleting an edge separating two different cells. When $s = 1$, the graph K' is a θ -curve. \square

Theorem 13. *Let F be a nonsingular fiber of the fibering $p: M^3 \xrightarrow{T^2} S^1$, and $K = P \cap F$. Then K contains a subgraph L that is a θ -curve. The number of pairwise nonisotopic θ -curves contained in K is finite.*

Proof. Combine Lemmas 7 and 8 and note that the number of all subgraphs of K is finite. \square

Let F_- and F_+ be two close nonsingular fibers and $L_- \subset K_-$ be a θ -curve in F_- . If the interval (t_-, t_+) contains no singular values of $p|_P$, then the graphs K_- and K_+ are isotopic, and K_+ contains a θ -curve L_+ isotopic to L_- . If there is a

singular value $t_0 \in (t_-, t_+)$, then K_+ may not contain a θ -curve isotopic to L_- , see, for example, Fig. 19, where a saddle Morse transformation occurs in a 2-component of P as $t = t_0$. By Theorem 13, K_+ still contains θ -curves. It is important that some of them are not too distant from L_- in the graph Γ , that is, can be obtained from L_- by a small number of flips.

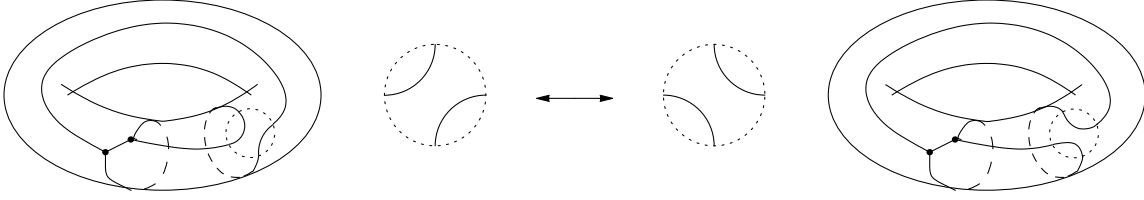


FIGURE 19. Transformation of L induced by a saddle point

Theorem 14. *Let both trivalent graphs $K_-, K_+ \subset T^2$ contain θ -curves and differ by one of the transformations shown on Figs. 15, 16, and 18. Then for any θ -curve $L_- \subset K_-$ there exists a θ -curve $L_+ \subset K_+$ such that $d(W_+, W_-) \leq 5$, where W_{\pm} is the hexagon corresponding to L_{\pm} and d is the distance in Γ , see Section 2.1.*

For a graph K , we can consider the set $w(K) \subset \Gamma$ of the hexagons in \mathbb{Z}^2 that correspond to all θ -curves contained in K . Theorem 14 proclaims that $d(w(K_-), w(K_+)) \leq 5$, where $d(X, Y)$ is the distance from a subspace X to a subspace Y of a metric space Γ defined by the formula $d(X, Y) = \max_{x \in X} \min_{y \in Y} d(x, y)$; note that $d(X, Y)$ need not be symmetric.

Proof. First, suppose that the whole graph $\text{lk}_P V$ lies in one hemisphere, where $V \in F_0$ is a singular point of $p|_P$ or a vertex. This happens in the situations of Fig. 18 a and of the leftmost pictures of Figs. 15 and 16. Then $U(V)$ contains only an isolated connected component of K_+ (or K_-), which has nothing to do with θ -curves in K_{\pm} . So the transformation of K arising from a singularity at V does not affect L_- , and the graph K_+ contains a θ -curve L_+ isotopic to L_- . Of course, in this case $d(W_+, W_-) = 0$.

In the cases represented by Fig. 18 b and by the second picture of Fig. 16, it is easy to see that the part of any θ -curve lying inside of any dotted circle is a part of an edge of the θ -curve (if nonempty), and the part of the θ -curve L_- lying outside of the dotted circle can be augmented with a path in K_+ lying inside of the dotted circle and homotopic to the path in K_- involved in the θ -curve L_- . Similar argument works for the case of Fig. 18 c, where the part of L_- lying inside of the dotted circle may be a tripod (a neighborhood of a trivalent vertex) or a part of an edge of L_- or the empty set. So, $d(W_+, W_-) = 0$ in all the cases where $\text{lk}_P V$ intersects the equator in less than four points.

The following Lemma is necessary to deal with the seven remaining cases (Fig. 18 d–g, two last pictures of Fig. 16, and the saddle transformation shown on Fig. 15, right).

Lemma 9. *Let $K_-, K_+ \subset T^2$ be trivalent graphs and $L_- \subset K_-$ be a θ -curve. Suppose that K_- differs from K_+ by one edge only and K_+ contains a θ -curve, too.*

1) *If K_+ is obtained from K_- by adding one extra edge, then it contains a θ -curve L_+ such that $d(W_+, W_-) = 0$.*

2) If K_+ is obtained from K_- by deleting an edge e , then it contains a θ -curve L_+ such that $d(W_+, W_-) \leq 1$.

Proof. The first statement is obvious, since we can put $L_+ = L_-$. In the second case, we also can put $L_+ = L_-$ unless $e \subset L_-$. Suppose that the edge e is a part of an edge l of L_- . Cut the torus T^2 along L_- into a hexagonal 2-cell H . The boundary ∂H contains two arcs $e_1 \subset l_1$, $e_2 \subset l_2$ arising from e , and two arcs $f_1 = FAB$, $f_2 = CDE$ complementary to the arcs $l_1 = FE$, $l_2 = BC$ that arise from l , see Fig. 20.

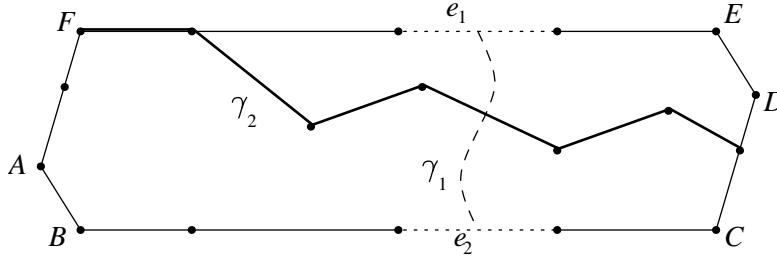


FIGURE 20. The hexagon $H = T^2 \setminus L_-$

There is the following alternative: either there exists a path γ_1 in $H \setminus K_+$ from the midpoint of e_1 to the midpoint of e_2 , or there exists a path γ_2 in K_+ from a point in f_1 to a point in f_2 ; this path γ_2 may include edges of K_+ belonging to $l \setminus e$. The first case is impossible, because the path γ_1 yields a nontrivial cycle in $T^2 \setminus K_+$; so we have the second case.

Put $L_+ = (L_- \setminus l) \cup \gamma_2$. Obviously, L_+ is a trivalent subgraph of K_+ with two vertices at the endpoints of γ_2 . It contains two edges of L_- different from l , which form a nontrivial cycle σ homotopic to γ_1 . Thus L_+ intersects any cycle $m\sigma + n\mu \in \pi_1(T^2)$ with $n \neq 0$, where μ is any cycle such that σ and μ generate $\pi_1(T^2)$. By construction, L_+ intersects any cycle homotopic to σ , too. Thus, L_+ is a θ -curve.

The path γ_2 considered above divides ∂H in two arcs. If the vertices A, D of H , which correspond to two different vertices of L_- (see Fig. 20), belong to the same arc, then L_+ is isotopic to L_- and $W_+ = W_-$. If they belong to different arcs, then $d(W_+, W_-) = 1$. \square

Let us return to the proof of Theorem 14. Consider the case of Fig. 18 e. Realize the graph K_- as the union of solid and dotted lines on the middle picture of Fig. 21. First, add the dashed edge to K_- . Then delete the dotted edge from the resulting graph. This results in K_+ . By Lemma 9, the first step does not affect any θ -curve in K_- , and the graph obtained after the second step contains a θ -curve L_+ such that $d(W_+, W_-) \leq 1$.

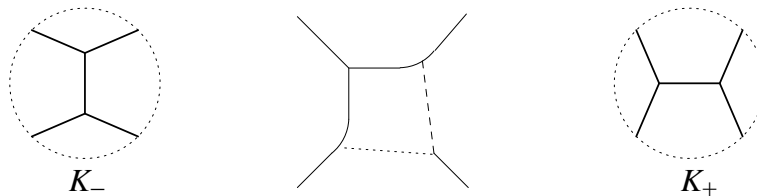


FIGURE 21. Step by step transformation of K_- to K_+

The same reasoning applies in all other cases. For instance, for the saddle transformation (see Fig. 15, right) we need first to add two extra edges that will be included in K_+ , and then delete two edges of $K_+ \setminus K_-$ from the graph obtained. Lemma 9 is applicable at each step, because any auxiliary graph contains either K_- (whenever an edge is added at the previous step) or K_+ (whenever an edge is deleted at the next step) and thus contains θ -curves. So for the saddle transformation we can guarantee that $d(W_+, W_-) \leq 2$ for some θ -curve $L_+ \subset K_+$. Figure 19 shows that this estimate is exact.

It is easy to see that in all remaining cases the graphs K_- and K_+ can be related by a sequence of m operations of adding an edge followed by n operations of deleting an edge, where $m, n \leq 5$. In fact, there are sequences giving the pairs (m, n) equal to $(2, 1)$ and $(4, 3)$ for the last two pictures of Fig. 16, $(2, 1)$ for the case of Fig. 18 d, $(3, 3)$ and $(5, 5)$ for the cases of Fig. 18 f and Fig. 18 g. By Lemma 9, there exists a θ -curve $L_+ \subset K_+$ such that $d(W_+, W_-)$ does not exceed the number of deletion steps, which is n for “left to right” (or “bottom to top”) transformations shown on Figs. 15, 16, and 18, and m for reverse transformations. In any case, this number is at most 5, which proves the Theorem. \square

Remarks. Theorem 14 gives an upper bound for $d(w(K_+), w(K_-))$. As we already saw (Fig. 19), this estimate is exact in the case of the saddle transformation. In fact, there are examples showing that the upper bound obtained in Theorem 14 is attainable for the third picture of Fig. 16, for the cases shown on Fig. 18 d, e, and for the “top to bottom” transformation of Fig. 18 f. In particular, the distance $d(w(K_+), w(K_-))$ is not symmetric for some graphs K_+ and K_- related by a transformation shown on Fig. 18 d or on the third picture of Fig. 16.

The estimate for $d(w(K_+), w(K_-))$ given in Theorem 14 exceeds 3 only in the case of Fig. 18 g and for the “right to left” transformation from the rightmost picture of Fig. 16. Even in these cases, no examples are known where $d(w(K_+), w(K_-)) > 3$. We believe that such examples do not exist. This would imply that the upper bound 5 in Theorem 14 can be replaced by 3; the existing examples show that it cannot be replaced by a number smaller than 3.

3.3. Proof of Theorem 12.

Let $t \in \mathbb{R}/\mathbb{Z}$ be a parameter on the circle $S^1 = \mathbb{R}/\mathbb{Z}$. Without loss of generality, it can be assumed that the fiber $F_0 = p^{-1}(0)$ contains no singular points of p .

The transversality hypothesis in Theorem 12 means that p has singular points neither inside of 2-components nor inside of edges of P . Thus the only singularities of $p|_P$ are the vertices of P ; we can assume that the vertices lie in pairwise different fibers. Let $t_1 < t_2 < \dots < t_n$ be their projections on S^1 (that is, the singular values of p); note that $n = c(P)$. For any $i = 1, \dots, n-1$ consider a nonsingular fiber $F_i = F_{(t_i+t_{i+1})/2}$. Also put $F_n = F_0$.

By virtue of Theorem 13, the graph $K_0 = P \cap F_0$ contains a finite number $N > 0$ of θ -curves. Let L_0 be one of them. By Theorem 14, we can construct step by step a sequence $L_i \subset K_i = P \cap F_i$, $i = 1, \dots, n$, of θ -curves such that $d(W_i, W_{i-1}) \leq 5$. This implies the inequality $d(W_n, W_0) \leq 5n$.

Recall that L_n is one of the N θ -curves contained in K_0 . First suppose that L_n coincides with L_0 . For the corresponding hexagons W_0 and W_n this means that $W_n = AW_0$; here A is the matrix of the monodromy operator \mathcal{A} in the basis where W_0 is the standard hexagon. Consequently, $d(W_0, W_n) = c(A) \geq c(\mathcal{A})$. Combining

this with the inequality $d(W_n, W_0) \leq 5n$, we get $c(\mathcal{A}) \leq 5n$ or, equivalently, $c(P) \geq c(\mathcal{A})/5$.

Note that there are at least two *loose* vertices of P , that is, vertices where the graph K undergoes the transformation shown on Fig. 18c. One of them corresponds to the minimum point of $\tilde{p}|_{\overline{D}}$ and another to the maximum point, where $\tilde{p}: \widetilde{M}(\mathcal{A}) \rightarrow \mathbb{R}$ is the function that covers p under the universal covering $\widetilde{M}(\mathcal{A}) \rightarrow M(\mathcal{A})$ and D is one of the preimages of the 3-cell $M(\mathcal{A}) \setminus P$ under this covering. Since loose vertices do not affect θ -curves and corresponding hexagons, we have $d(W_n, W_0) \leq 5(n-k)$, where $k \geq 2$ is the number of the loose vertices of P . This proves the inequality $c(P) \geq \frac{1}{5}c(\mathcal{A}) + k \geq \frac{1}{5}c(\mathcal{A}) + 2$ in the case $L_n = L_0$.

Now suppose that L_n does not coincide with L_0 . Consider a sequence $L_{n+1}, L_{n+2}, \dots, L_{Nn}$, where $L_i \subset K_{i'} = P \cap F_{i'}$ with $i' \equiv i \pmod{n}$, and $d(W_i, W_{i-1}) \leq 5$ for all $i = 1, \dots, Nn$; we also assume that $W_i = W_{i+1}$ whenever the vertex lying between $F_{i'}$ and $F_{i'+1}$ is a loose vertex. Since the graph K_0 contains only N different θ -curves, we have $L_{sn} = L_{tn}$ for some s, t such that $0 \leq s < t \leq N$. Then, as above, we have $c(\mathcal{A}^{t-s}) \leq d(W_{sn}, W_{tn}) \leq 5(t-s)(n-k)$, because $W_{tn} = \mathcal{A}^{t-s}W_{sn}$ and there are $k \geq 2$ loose vertices. By Theorem 8, we have $c(\mathcal{A}^{t-s}) = (t-s)c(\mathcal{A})$ whenever monodromy operator \mathcal{A} is non-periodic. Thus $5(t-s)(c(P) - k) \geq (t-s)c(\mathcal{A})$, i.e., $c(P) \geq \frac{1}{5}c(\mathcal{A}) + k \geq \frac{1}{5}c(\mathcal{A}) + 2$. The proof of Theorem 12 is complete. \square

Remark. It was shown in [21] that a spine P can be deformed so that the only transformations of the level sets are two Morse transformations (see Fig. 15), the transformations shown on the second picture on Fig. 16, and flips, see Fig. 18d. (In fact, this statement is proved in [21] for functions with values in the interval $[0, 1]$, but the proof applies to S^1 -valued functions, too.) Note that a flip in K induces at most one flip in L and a saddle transformation induces at most two flips, while two other transformations do not require flips at all. This yields the estimate $c(P) \geq c(\mathcal{A}) + 2 - 2s$ (where s is the number of saddle points of p), which holds for arbitrary spine of $M(\mathcal{A})$, implying $c(M(\mathcal{A})) \geq c(\mathcal{A}) + 2 - 2s$. However, it is unlikely that there exists any lower bound for s .

§4. MAIN CONJECTURES

4.1. Triangulations of T^2 coming from spines.

Let us recall the idea of the proof of Theorem 12. Given a spine P of $M^3 = M^3(\mathcal{A})$, suppose that we can assign some object σ_t to each fiber F_t containing no vertices of P in such a way that

- 1) σ_t is an element of some metric space (Σ, d) ;
- 2) there is an action of the group $\text{SL}(2, \mathbb{Z})$ on the set Σ , and monodromy \mathcal{A} takes σ_t to $\sigma_{t+2\pi} = \mathcal{A}\sigma_t$;
- 3) $d(\sigma_t, \mathcal{A}\sigma_t) \geq c(\mathcal{A})$;
- 4) as t varies, σ_t remains unchanged until F_t encounters a vertex $V \in P$;
- 5) if there is exactly one vertex of P between F_{t+} and F_{t-} (we can assume that vertices of P lie in pairwise different fibers), then $d(\sigma_{t+}, \sigma_{t-}) \leq 1$.

Then the number of vertices of P is at least $c(\mathcal{A})$. This can be followed by an argument that shows that this number is in fact at least $c(\mathcal{A}) + k$, where k is some positive integer smaller than 6.

In the proof of Theorem 12, σ_t is an isotopy class of θ -curves in F_t , Σ is the vertex set of the trivalent graph Γ described in Section 2.1, d is the distance in Γ ,

and $k = 2$. To preserve property 4, we had to restrict ourselves to spines transversal to the fibers. Still, we got $d(\sigma_{t+}, \sigma_{t-}) \leq 5$ instead of property 5; this leads to an annoying factor of $1/5$ in Theorem 12.

In this section we present another construction for σ_t ; namely, it will be some class of 2-chains representing the fundamental class $[F_t]$. This approach applies to all special spines P of $M(\mathcal{A})$ (recall that, by Theorem 2, any minimal spine of $M(\mathcal{A})$ is a special one), not only to those transversal to the fibers, and properties 1, 2, 4, and 5 hold, while property 3 remains unproved.

Given a special spine P of M^3 , by P' denote a triangulation of M^3 dual to P . Its tetrahedra correspond to the vertices of P (see Fig. 1 f), triangles correspond to the edges of the singular graph SP (that is, to the triple lines) of the spine P , edges of P' are dual to the 2-cells of P , and the only vertex of P' is located somewhere inside of the 3-cell $M \setminus P$. Here and below we abuse the word “triangulation”: the intersection of two simplices (if nonempty) is a subset of the sets of their faces of smaller dimensions, but the cardinality of this subset may exceed 1.

Let $f: \mathbb{R}^1 \rightarrow S^1$ be the standard covering defined by the rule $f(t) = t \bmod 2\pi$. By $\tilde{f}: \tilde{M}^3 \rightarrow M^3$ denote the covering such that $\tilde{f}_*(\pi_1(\tilde{M}^3)) = i_*(\pi_1(F))$, where $i: F \rightarrow M^3$ is the embedding map of some fiber F into M^3 . Then a projection $\tilde{p}: \tilde{M} \rightarrow \mathbb{R}^1$ can be defined by the relation $f \circ \tilde{p} = p \circ \tilde{f}$. By $F_t \subset \tilde{M}$ denote the fiber $\tilde{p}^{-1}(t)$, where $t \in \mathbb{R}^1$. Put $\tilde{P} = \tilde{f}^{-1}(P)$ and $\tilde{P}' = \tilde{f}^{-1}(P')$. Note that \tilde{P} is a spine of \tilde{M} punctured infinitely many times (at all preimages of the vertex V' of P') and \tilde{P}' is a triangulation of \tilde{M} . Furthermore, \tilde{M} is nothing but $T^2 \times \mathbb{R}^1$ equipped with the deck transformation $(x, t) \mapsto (\mathcal{A}x, t + 2\pi)$, where $x \in T^2$ and $t \in \mathbb{R}^1$, and both \tilde{P} and \tilde{P}' are invariant under the deck transformation. Fix a cartesian product structure on \tilde{M} and define the forgetting projection $j: \tilde{M} \rightarrow T^2$ by $j(x, t) = x$.

By M_0 , respectively, \tilde{M}_0 , denote the complement to the vertices of P in M , respectively, to the vertices of \tilde{P} in \tilde{M} . It is easy to see that there is a strict deformation retraction $r: M_0 \rightarrow \text{sk}_2 P'$, which is covered by a strict deformation retraction $\tilde{r}: \tilde{M}_0 \rightarrow \text{sk}_2 \tilde{P}'$. If a fiber F_t contains no vertices of \tilde{P} , the image $\tilde{r}(F_t) \subset \text{sk}_2 \tilde{P}'$ defines a 2-cycle $c_t^2 \in Z_2(\tilde{P}')$. The group $Z_2(\tilde{P}')$ is generated by the 2-cells of P' , so we have $c_t^2 = \sum \alpha_k(t) \tau_k^2$, where $\{\tau_k^2\}$ is the set of the 2-cells of P' .

The coefficients $\alpha_k(t)$ are equal to the intersection indices of F_t with the oriented edges e_k of \tilde{P} corresponding to the (oriented) 2-cells τ_k^2 . Let A_k and B_k be the endpoints of an edge e_k oriented from A_k to B_k . If A_k and B_k lie both below or both above F_t (that is, if $(\tilde{p}(A_k) - t)(\tilde{p}(B_k) - t) > 0$), then $\alpha_k(t) = 0$; otherwise, $\alpha_k(t) = \text{sign}(\tilde{p}(B_k) - \tilde{p}(A_k)) = \pm 1$. Obviously, only finite number of the coefficients can be nonzero, and property 4 holds for all $\alpha_k(t)$, whence for c_t^2 as well.

Let us consider a 2-chain $j_*(c_t^2)$. By construction, it is a cycle and its homology class is $[T^2]$ (because $j_*(c_t^2) = j_*(F_t)$). All triangles that constitute $j_*(c_t^2)$ have all their vertices mapped to $j(V')$ (where V' is the vertex of P'). Let σ_t be the linearization of $j_*(c_t^2)$, that is, the 2-chain obtained from $j_*(c_t^2)$ by replacing the characteristic mappings of all triangles by homotopic (with fixed vertices) linear mappings; a mapping of a triangle to the torus is linear if it is a linear mapping to the plane followed by the projection $\mathbb{R}^2 \rightarrow T^2$. Roughly speaking, we define Σ as the set of all linear 2-cycles in T^2 that are homological to $[T^2]$ and have the only vertex, which is placed at $j(V')$. Accurately speaking, any element of Σ is a (homological to $[T^2]$) cycle $\sum \beta_k \Delta_k$, where the coefficients β_k are integers, all but finite number of them are zero, and $\{\Delta_k\}$ is the set of projections (from \mathbb{R}^2

to T^2) of all different oriented triangles with vertices in \mathbb{Z}^2 ; the triangles are equal if they come from the same edge e_k of \tilde{P}' , so even geometrically coinciding triangles may be different in this sense. This difference will be exploited later, but unless otherwise stated, we identify all coinciding triangles. Degenerated triangles (those with three vertices along a line) are allowed, and their orientation is, as usually, a cyclic ordering of their vertices. The metric on Σ will be constructed later.

Obviously, we have $\sigma_t \in \Sigma$. Further, the group $\mathrm{SL}(2, \mathbb{Z})$ acts on Σ in a natural way, and $\sigma_{t+2\pi} = \mathcal{A}\sigma_t$, where \mathcal{A} is the monodromy of the fibration $p: M^3 \rightarrow S^1$. Thus properties 2 and 4, see above, are satisfied; further, property 1 is satisfied for any choice of metric d .

To find the difference between σ_{t_+} and σ_{t_-} (where the interval (t_-, t_+) contains the projection $\tilde{p}(V)$ of exactly one vertex V of \tilde{P}), consider the difference between F_{t_+} and F_{t_-} . It is easy to see that F_{t_+} can be obtained as a connected sum of F_{t_-} and a small two-dimensional sphere $S^2(V)$ centered at V . Thus σ_{t_+} is obtained from σ_{t_-} by adding $j_*(\tilde{r}(S^2(V)))$, that is, by one of the two-dimensional Pachner moves, see [20] and §2.3 above. Indeed, there are four edges of \tilde{P} emanating from V . Suppose that they are all different, that is, they are not loops. Then $\tilde{r}(S^2(V))$ is the boundary of a tetrahedron triangulated into four triangles τ_k^2 , from which 0 to 4 can annihilate with triangles that constitute $\tilde{r}(F_{t_-})$. So σ_{t_+} is obtained from σ_{t_-} in one of the following five ways:

- 0) by adding the projection of the boundary of some tetrahedron, that is, by adding triangles BCD , CAD , ABD , and BAC (mind the orientations!), or
- 1) by replacing a triangle ABC of σ_{t_-} by three triangles ABD , BCD , and CAD , or
- 2) by replacing two triangles ABC and ACD of σ_{t_-} (the segment AC contributes to both their boundaries, but with opposite signs) by the triangles ABD and BDC , or
- 3) by replacing three triangles ABD , BCD , and CAD by a single triangle ABC , or
- 4) by erasing four triangles BCD , CAD , ABD , and BAC .

Of course, moves 3 and 4 may not be applicable to arbitrary $\sigma_t \in \Sigma$.

If one of the edges of \tilde{P} incident to V is a loop, then two of the triangles that form $\tilde{r}(S^2(V))$ annihilate with one another, and it is easy to see that $\sigma_{t_+} = \sigma_{t_-}$.

Consider a graph with vertices at the elements of Σ and edges corresponding to the moves described above. Let d be the distance function on this graph. This completes the construction of the metric space (Σ, d) . Property 5 now holds by construction. Recall that properties 1, 2, and 4 hold as well. Property 3 is a conjecture. If it holds, then the inequality $c(M(\mathcal{A})) \geq c(\mathcal{A})$ follows immediately.

This conjecture is supported by the following observation. For any triangle of σ_t , draw three segments connecting its baricenter with the midpoints of its sides. All the tripods obtained in this way form a graph K . In some simple cases (for example, for spines constructed in Section 2.3), K is a trivalent graph embedded into T^2 . The Pachner moves 0 and 4 affect K according to Fig. 18a (“left to right” transformation of Fig. 18a for move 0 and “right to left” for move 4); moves 1 and 3 affect K according to Fig. 18c; finally, moves 2 correspond to flips, see Fig. 18e. It is possible now to choose a θ -curve in K and apply the argument of §3, taking into account that moves of Figs. 18a and 18c do not affect the isotopy classes of θ -curves and any flip in the graph K implies at most one flip of any θ -curve in it.

However, in the general case the graph K neither is embedded into the torus nor is trivalent, because there may be 4, 6 etc. triangles having the same edge in common, even for spines transversal to the fibers.

There is a more geometrical reformulation of this conjecture. For a chain $c = \sum_i r_i \tau_i$, where the sum is finite, $r_i \in \mathbb{Z}$, and the τ_i are singular simplices, put $\|c\| = \sum_i |r_i|$, and consider the minimum value $l^1(\mathcal{A})$ of $\|c\|$ over all 3-chains in $T^2 \times I$, $I = [0, 2\pi]$, representing the fundamental class $[T^2 \times I]$ and satisfying the condition $\sigma_{2\pi} = \mathcal{A}\sigma_0$, where 2-chains $\sigma_{2\pi}$ and σ_0 are the intersections of the chain c with $T^2 \times \{2\pi\}$ and $T^2 \times \{0\}$, and $\mathcal{A}(x, 0) = (\mathcal{A}x, 2\pi)$ for $(x, 0) \in T^2 \times \{0\}$. It is easy to see that $l^1(\mathcal{A}) \leq c(\mathcal{A})$. The conjecture can be formulated as follows: $l^1(\mathcal{A}) = c(\mathcal{A})$. The definition of $l^1(\mathcal{A})$ is similar to Gromov's definition of the simplicial l^1 -norm (with some boundary conditions imposed), see [10]. However, with the original definition by Gromov, we have $\|M^3(\mathcal{A})\|_{l^1} = 0$. Also see the concluding remarks in [26].

In fact, the estimate $c(M(\mathcal{A})) \geq c(\mathcal{A})$ can be deduced from Conjecture 3 (see below), which is weaker than the conjecture above. Recall that the triangles of σ_t are "marked" by the corresponding edges of \tilde{P} . It may happen that two different edges contribute two equal but oppositely oriented triangles to σ_t . Let us prohibit cancellation of the triangles in these cases. Then we get the set Σ' of 2-cycles homological to $[T^2]$ such that any triangle is marked by two nonnegative integers, say, (m, n) , which means that σ' contains m positively oriented copies of this triangle and n negatively oriented copies of it. A natural mapping $s: \Sigma' \rightarrow \Sigma$ is defined by replacing a mark (m, n) by the coefficient $m - n$. To define a distance d' on Σ' , we follow the construction of the distance function d ; however, we prohibit moves 0 and 4; only moves 1, 2, and 3 are allowed. Obviously, the inequality $d'(\sigma_1, \sigma_2) \geq d(s(\sigma_1), s(\sigma_2))$ holds. Note also that $s(\mathcal{A}\sigma) = \mathcal{A}s(\sigma)$, where $\sigma \in \Sigma'$ and $\mathcal{A} \in \text{SL}(2, \mathbb{Z})$. So the following statement is weaker than the conjecture above (which states that property 3 holds).

Conjecture 3. *For all $\sigma \in \Sigma'$ and $\mathcal{A} \in \text{SL}(2, \mathbb{Z})$, the inequality $d'(\mathcal{A}\sigma, \sigma) \geq c(\mathcal{A})$ holds.*

However, this weaker conjecture still implies the inequality $c(M(\mathcal{A})) \geq c(\mathcal{A})$.

Theorem 15. *Conjecture 3 implies the estimate $c(M(\mathcal{A})) \geq c(\mathcal{A})$.*

Proof. Let us say that a vertex V of \tilde{P} is *maximal*, respectively, *minimal*, if the endpoints of all four edges going from V are not above, respectively, not below V (for $A, B \in \tilde{M}$, we say that A lies above B if $\tilde{p}(A) > \tilde{p}(B)$; further, we assume that $\tilde{p}(A) \neq \tilde{p}(B)$ whenever vertices A and B are different). Maximal and minimal vertices are also called *critical*. If a vertex $V \in \tilde{P}$ is maximal (respectively, minimal, critical), then its projection $\tilde{f}(V) \in P$ is said to be a *maximal* (respectively, *minimal*, *critical*) vertex of P . Obviously, maximal (minimal, critical) vertices of P are well-defined, because for any vertex $V \in P$ all its preimages $\tilde{f}^{-1}(V)$ are or are not critical (maximal, minimal) vertices of \tilde{P} simultaneously.

First, suppose that there are no critical vertices. Then any vertex of \tilde{P} induces a change of $\sigma_t \in \Sigma'$ by one of the moves 1, 2, 3. (However, it can happen that the chain $s(\sigma_t) \in \Sigma$ undergoes the move 0 or 4. That is why we had to introduce Σ' instead of Σ .) It follows that in this case $d(s(\sigma_t), s(\sigma_{t+2\pi})) = d'(\sigma_t, \sigma_{t+2\pi})$, so Conjecture 3 implies property 3 (see the beginning of §4), and the spine P contains at least $c(\mathcal{A})$ vertices.

Suppose that \tilde{P} contains critical vertices, but there is an isotopy of the embedding of P in M^3 such that the deformed spine P_1 yields the deformed covering spine \tilde{P}_1 with neither minimal no maximal vertices. Then the number of the vertices of P_1 is at least $c(\mathcal{A})$ (provided that Conjecture 3 is true); obviously, P and P_1 have the same number of vertices.

The notion of peripheric edge is necessary for the case of arbitrary spines.

Definition 13. An edge e of the singular graph SP of a spine P is said to be *peripheric* if there exists a vertex V of P with the following property: if a cycle $\gamma \subset SP$ contains e and $p_*(\gamma) \neq 0$ (here p is the projection $M^3 \rightarrow S^1$), then every connected component of $\gamma \setminus V$ containing e is a loop γ' (with endpoints at V) such that $p_*(\gamma') = 0$; in the other words, the map p_* restricted to the closure of the connected component of $SP \setminus V$ containing e is trivial. Edges that are not peripheric are called *regular*.

For example, a loop e is peripheric if and only if $p_*(e) = 0$. Unlike the property of a vertex to be critical, the property of an edge to be peripheric is an isotopy invariant.

Lemma 10. *If P contains no peripheric edges, then it is isotopic to a spine with no critical vertices.*

Proof. Take any edge $e_1 \in P$. Since e_1 is a regular edge, there exists a cycle $\gamma_1 \subset SP$ such that $e_1 \subset \gamma_1$, $p_*(\gamma_1) \neq 0$, and γ_1 does not pass twice through any vertex. Then there exists a spine P_1 isotopic to P such that the projection $\tilde{p}: \tilde{M} \rightarrow \mathbb{R}^1$ restricted to a connected component $\tilde{\gamma}'_1$ of $\tilde{f}^{-1}(\gamma'_1)$, where $\gamma'_1 \subset P_1$, takes the sequence of the vertices of $\tilde{\gamma}'_1$ to a strictly monotone sequence $\{a_i \mid i \in \mathbb{Z}\}$ of real numbers; here γ'_1 is obtained from γ_1 under an isotopy that takes P to P_1 . To construct P_1 , fix a monotone sequence $\{a_i\}$ satisfying the condition $a_{i+m} = a_i + 2\pi k$, where m is the length of γ_1 and $k \in \mathbb{Z} = \pi_1(S^1)$ is equal to $p_*(\gamma_1)$. Let $\{b_i \mid i \in \mathbb{Z}\}$ be the sequence of the projections $\tilde{p}(V_i)$, where the V_i are consecutive vertices of $\tilde{\gamma}_1$. The differences $c_k = a_k - b_k$ form an m -periodic sequence. Let $\{\delta_i \mid i = 1, \dots, m\}$, be a family of arcs in \tilde{M} such that the endpoints of δ_i are V_i and V'_i , where $\tilde{p}(V'_i) = a_i$ and the δ_i do not intersect \tilde{SP} ; suppose also that the projections $\tilde{f}(\delta_i)$ do not intersect one another. It is easy to see that such a family exists, and that it is possible to get a spine P_1 with the properties described above by an isotopy of the mapping $\text{id}: M^3 \rightarrow M^3$, fixed outside of a small collar neighborhood of $\bigcup_{i=1}^m s(\delta_i)$.

Now any vertex $V \in \gamma_1$ is not critical. If there remain critical vertices, take an edge $e_2 \in P_1$ incident to one of them and a cycle $\gamma_2 \subset SP_1$ containing e_2 such that $p_*(\gamma_2) \neq 0$ and γ_2 does not pass twice through any vertex. If $\gamma_2 \cap \gamma_1 = \emptyset$, repeat the construction above. Otherwise, take the connected component γ_2° of $\gamma_2 \setminus \gamma_1$ that contains e_2 , and move the inner vertices of this component as above, but do not move its endpoints. In the spine P_2 obtained after this step, any vertex $V \in \gamma_1 \cup \gamma_2^\circ$ is not critical. Indeed, any vertex of γ_2° is a noncritical vertex of P_2 by construction, and any vertex $V \in \gamma_1$ has neighboring (in γ_1) vertices above and below V in P_1 ; P_2 inherits this property from P_1 , because the isotopy converting P_1 into P_2 does not affect γ_1 .

If there still remain critical vertices, repeat the construction described above. Then we get a spine P_3 isotopic to P_2 . The set of the vertices of $\gamma_1 \cup \gamma_2^\circ \cup \gamma_3^\circ$, which certainly are not critical in P_3 , is larger than the similar set $\gamma_1 \cup \gamma_2^\circ$ in P_2 . Hence,

in a finite number of steps we obtain a spine P_k isotopic to P and having no critical vertices. \square

According to Lemma 10, it follows from Conjecture 3 that any spine of $M(\mathcal{A})$ without peripheric edges contains at least $c(\mathcal{A})$ vertices.

The degree of any vertex $V \in SP$ equals four. Suppose that there are k peripheric and $4 - k$ regular half-edges incident to V (we consider half-edges, because the graph SP may contain loops). Note that the number k is even, thence is equal to 0, 2 or 4. Indeed, if there is a regular edge e incident to V (that is, if $k < 4$), then there is a cycle γ consisting of regular edges and passing through V , so in this case the number of regular half-edges incident to V is at least 2, and we have $k \leq 2$. On the other hand, if $k = 1$, consider the connected component of $SP \setminus V$ containing the peripheric edge e incident to V . Let it contain m edges and n vertices different from V . Then the number of half-edges in this component is equal to $2m$ and to $4n + 1$ simultaneously, which is impossible.

Let P be an arbitrary spine of $M(\mathcal{A})$. Let us say that a vertex is *regular* if it is incident to regular edges only, *semiregular* if it is incident to two peripheric and two regular half-edges, and *peripheric* if it is incident to peripheric edges only. Following the proof of Lemma 10, replace P by an isotopic spine P_1 such that all regular and semiregular vertices of P_1 are not critical. For any semiregular vertex V of P_1 , by $b(V)$ denote the connected component of $SP \setminus V$ that consists of peripheric edges only (i.e., lies “behind V ”). Following the proof of Lemma 10, replace P_1 by a spine P_2 such that P_2 is isotopic to P_1 , the isotopy between P_1 and P_2 does not affect regular edges, $p(V_i) = p(V)$ for all semiregular vertices $V \in P_2$ and all peripheric vertices $V_i \in b(V)$, and $p(e)$ is a zero-homotopic loop in S^1 for any peripheric edge e .

Consider 2-cycles $\sigma_t \in \Sigma'$ constructed from the spine P_2 . If there is exactly one regular vertex between F_{t+} and F_{t-} , then σ_{t+} is obtained from σ_{t-} by one of the moves 1, 2, 3, so $d'(\sigma_{t+}, \sigma_{t-}) = 1$.

Suppose that there is exactly one semiregular vertex $V \in \tilde{P}_2$ (together with all peripheric vertices of $b(V)$) between F_{t+} and F_{t-} . Compare σ_{t-} and σ_{t+} . The former cycle consists of the triangle corresponding to the regular edge e_- of \tilde{P}_2 incident to V such that the other endpoint of e_- lies below V , and of some other triangles. The latter cycle includes the triangle that corresponds to the other regular edge e_+ of \tilde{P}_2 incident to V ; other triangles that constitute σ_{t+} are the same as in σ_{t-} . Note that all triangles but one of a 2-cycle $\sigma \in \Sigma'$ determine the remaining triangle Δ uniquely: directions of its sides are those of the boundary of the 2-chain formed by the remaining triangles and thus denote Δ up to dilatation with the coefficient ± 1 , and this coefficient is uniquely determined by the condition that σ is homological to $[T^2]$; informally, Δ is the difference between $[T^2]$ and the 2-chain formed by the other triangles of σ . So in this case we have $\sigma_{t+} = \sigma_{t-}$.

We have shown that regular vertices of P_2 shift σ_t by distance 1, while semiregular and peripheric vertices do not affect σ_t at all. Recall that regularity, semiregularity, and periphericity are preserved under isotopy. Consequently, Conjecture 3 implies that the number of regular vertices of an arbitrary spine P of $M(\mathcal{A})$ is at least $c(\mathcal{A})$, so the number of all vertices of P is greater than or equal to $c(\mathcal{A})$, too. This completes the proof of Theorem 15. \square

Finally, let us provide a sketchy explanation (or, maybe, rather motivation) for the “+5” summand in Conjecture 2. Suppose that there is an edge $e \in P'$ of

the triangulation P' dual to a given spine P such that $p_*(e) = \pm 1$ in the group $\pi_1(S^1) = \mathbb{Z}$. Note that the cartesian product structure on \widetilde{M} and the corresponding projection $j: \widetilde{M} \rightarrow T^2$ are not uniquely defined; in our case we can choose any element of $\mathbb{Z}^2 = \pi_1(T^2)$ to be the image $j_*(e)$. Choose j so that $j_*(e) = 0$. Cycles σ_t can include degenerated triangles with an edge $j_*(e)$, which are projections of the triangles of \widetilde{P}' incident to e . However, if vertices A and B of a triangle ABC coincide, the triangle ABC cannot be distinguished from BAC , which is equal, on the other hand, to ABC with the opposite orientation, so such triangles can be ignored in σ_t . It can be easily shown that Pachner moves 1, 2, and 3 involving triangles of this type do not affect σ_t and thus require k “additional” vertices of P , where k is the number of triangles of \widetilde{P}' incident to e , that is, the number of sides of the 2-cell of \widetilde{P} dual to $e \subset \widetilde{P}'$. We can assume P to be a minimal spine of $M(\mathcal{A})$. In this case P is a special spine, and, according to §1.2, it contains $n + 1$ 2-cells, where $n = c(M^3)$ is the number of its vertices. The total number of the vertices of 2-cells equals $6n$, so the average number of sides for a 2-cell of P is $\frac{6n}{n+1} > 5$ (because $n \geq 6$). Thus it is reasonable to expect at least 5 “spare” vertices, in addition to $c(\mathcal{A})$ vertices that do change σ_t .

Of course, it can happen that there are no edges e such that $p_*(e) = \pm 1$. However, there always exists an edge e such that $p_*(e) = m > 0$. Then the argument above can be applied to an m -fold covering $\widetilde{M}_m = M(\mathcal{A}^m)$ of $M(\mathcal{A})$. A special spine P of $M(\mathcal{A})$ (punctured once) is covered by a special spine \widetilde{P}_m of \widetilde{M}_m with m punctures, and we can destroy $m - 1$ 2-cells of \widetilde{P}_m in order to get a special spine of \widetilde{M}_m punctured only once. Apart from this, we destroy an m th 2-cell of \widetilde{P}_m by the process described in the previous paragraph. Consequently, we obtain vertices of m 2-cells of \widetilde{P}_m as additional, spare vertices, and $c(\mathcal{A}^m) = mc(\mathcal{A})$ (see Theorem 8) vertices of \widetilde{P}_m affecting σ_t ; obviously, the number of vertices of \widetilde{P}_m is equal to m times the number of the vertices of P . Unfortunately, the “averaging argument” provides no proof for the claim that, by destroying m 2-cells during the above-described process, we can destroy at least $5m$ vertices (in fact, $4m + 1$ would already suffice), even though there is an essential ambiguity in the choice of 2-cells to be discarded. Because of pseudominimality of P , all boundary curves of 2-cells are not short, see [14] and the beginning of §2.3 above, but this does not readily yield $4m$ of $4m + 1$ spare vertices required, since *a priori* the boundary curve of a 2-cell can visit some vertices more than once. It is only possible to prove with this approach that $2m$ vertices are destroyed when m 2-cells are discarded, and this results in 2 “additional” vertices of P , compare with Theorem 12.

4.2. Can hyperbolic geometry help?

Hyperbolic plane H^2 can be a natural receptacle for various metric spaces Σ introduced in §4.1, because H^2 is the Teichmüller space of T^2 . Being a subgroup of $\mathrm{SL}(2, \mathbb{R})$, the group $\mathrm{SL}(2, \mathbb{Z})$ acts on H^2 (modelled on the upper half-plane) by the rule

$$\mathcal{A}(z) = \frac{az + b}{cz + d}, \quad \text{where } \mathcal{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \quad \text{and} \quad \mathrm{Im} z > 0; \quad (4)$$

the kernel of this action is $\{\pm I\}$, so formula (4) defines an action of the modular group $\mathrm{SL}(2, \mathbb{Z})/\{\pm I\}$ on H^2 .

Consider the ideal triangle in H^2 with vertices at 0, 1, and ∞ . Take its mirror images in its sides. This gives the triangles $(-1, 0, \infty)$, $(0, 1/2, 1)$, and $(1, 2, \infty)$, where (a, b, c) denotes the ideal triangle with vertices a , b , and c . On the next step, construct the images of the triangles obtained in the previous step under reflections in their sides that are not sides of triangles obtained earlier. Continuing this way, we get a tessellation of H^2 into equal ideal triangles. It is called the *Farey tessellation*. Note that the modular group action (4) preserves it.

Consider a black-and-white coloring of triangles of the Farey tessellation such that neighboring triangles are of different colors. The only epimorphism of the modular group to \mathbb{Z}_2 discussed in Theorem 6 can be defined as follows: $\varphi(g) = 0$ if g preserves the coloring and $\varphi(g) = 1$ if g reverses it.

The following lemma (which certainly is well known though I did not find a reference) reveals the connection between the Farey tessellation and the Farey series.

Lemma 11. *Let r/s be the third vertex of a triangle Δ_1 obtained by the reflection in the side $(m/n, p/q)$ of a triangle Δ obtained on the previous step. Then $\frac{r}{s} = \frac{m+p}{n+q}$.*

Proof. The proof is by induction over the number of reflections, counting from the “root” triangle $\Delta_r = (0, 1, \infty)$. By the induction hypothesis, the third vertex of Δ is $\frac{m-p}{n-q}$. A direct calculation of the cross-ratio of the quadruple $(\frac{m}{n}, \frac{m+p}{n+q}, \frac{p}{q}, \frac{m-p}{n-q})$ shows that the points $\frac{m-p}{n-q}$ and $\frac{m+p}{n+q}$ are symmetric in the line $(\frac{m}{n}, \frac{p}{q})$. \square

A line $(m/n, p/q)$, where $m, n, p, q \in \mathbb{Z}$, occurs in the Farey tessellation if and only if $mq - np = \pm 1$ (we express ∞ as $1/0$). The “only if” statement is proved by induction on the number of reflections, as above. To prove the “if” statement, read Chapter III of [11] or note that m/n and p/q either are both nonnegative or both nonpositive, since otherwise $|mq - np| > 1$. Hence, the line $(m/n, p/q)$ does not intersect the line $(0, \infty)$. Any other line l of the Farey tessellation can be taken to $(0, \infty)$ by a sequence of reflections in its neighboring lines followed by a rotation of the triangle $(0, 1, \infty)$. Let this isometry of H^2 take the line $(m/n, p/q)$ to $(m'/n', p'/q')$, then $|m'q' - n'p'| = |mq - np| \neq 1$, so the line $(m/n, p/q)$ does not intersect l as well. Then it cannot pass through an interior point of any triangle of the tessellation, and the only chance for it to be contained somewhere in H^2 is to be one of the lines of the tessellation. For a constructive proof of the “if” statement, the Euclid algorithm should be applied to one of the pairs $(m+p, n+q)$ and $(m-p, n-q)$, compare with the proof of Theorem 2 in [2].

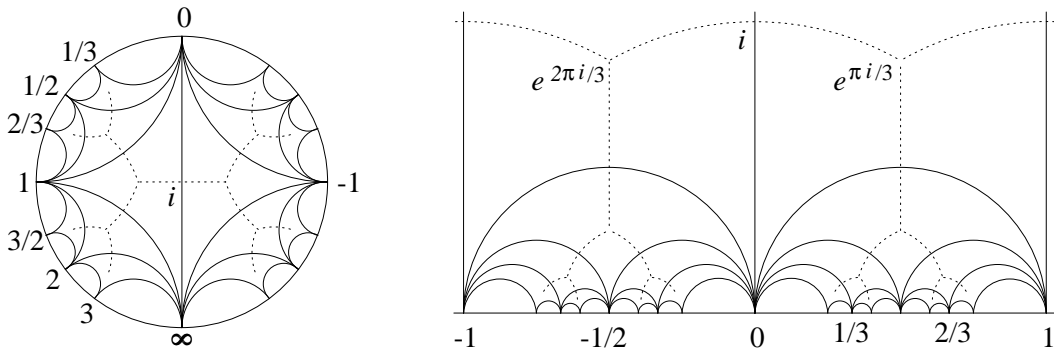


FIGURE 22. Farey tessellation of hyperbolic plane

Figure 22 represents the Farey tessellation of H^2 for the circle and half-plane models of hyperbolic plane. For more information on the Farey tessellation see [23].

Recall that isotopy classes of θ -curves correspond to admissible hexagons, see [2] or §2.1 above. There is a natural correspondence between admissible hexagons and triangles of the Farey tessellation: a hexagon W with vertices $\pm(m, n)$, $\pm(p, q)$, and $\pm(r, s)$ corresponds to an ideal triangle $\Delta = (m/n, p/q, r/s)$. For example, the standard hexagon W_0 corresponds to $\Delta_0 = (\infty, -1, 0)$. It is easy to check that a centrally symmetric hexagon W is admissible if and only if Δ is one of the triangles that form the Farey tessellation: both conditions are equivalent to the equations $|mq - np| = |ps - qr| = |rn - sm| = 1$. Moreover, flips of admissible hexagons correspond to reflections in sides of the corresponding triangles. Consequently, the graph Γ introduced in §2.1 is nothing but the dual graph of the tessellation; by the way, now it is obvious that Γ is a tree.

To embed Γ in H^2 in the most symmetric way, let its vertices be the centers of incircles² of corresponding ideal triangles. The incircles of two triangles having a common side touch this side at the same point, because the whole picture is invariant under reflection in this side. By a straightforward calculation, we obtain that the radius of the incircle of an ideal triangle equals $\frac{\ln 3}{2}$. So the hyperbolic length of any edge of $\Gamma \subset H^2$ is equal to $\ln 3$. Angles between edges at any vertex are equal to $2\pi/3$ because of symmetry. This metric information completely defines the embedding of Γ in H^2 .

The embedding $\Gamma \hookrightarrow H^2$ described above (and represented on Fig. 22 by dotted lines) makes the Farey tessellation coincide with the Voronoi partition of H^2 with respect to the vertices of Γ : each ideal triangle $\Delta(w)$ containing a vertex $w \in \Gamma$ coincides with the set of points $x \in H^2$ such that the vertex of Γ closest to x is w . Moreover, for any point $x \in \Delta(w)$ the point $y \in \Gamma$ closest to x lies in $\Delta(w)$, too. The graph Γ is the one-dimensional skeleton of the Delaunay decomposition of H^2 dual to the Farey tessellation (considered as a Voronoi partition).

There are two different distance functions on Γ : the hyperbolic distance d_H between its vertices and the “counting” distance d_c , which is equal to the number of edges in the shortest path between vertices. The distance d_c also can be defined as the number of the Farey tessellation lines that intersect the geodesic line between two given vertices; here vertices can be replaced by arbitrary points in the corresponding triangles. This implies the following statement.

Theorem 16. *For any point $z \in H^2$ not belonging to the Farey tessellation lines, we have $d_c(\mathcal{A}z, z) = c(\mathcal{A})$. \square*

Conjecture 4. *There exists some natural construction of a metric space (Σ, d) , where its elements $\sigma_t \in \Sigma$ are closely related to triangles in the Farey tessellation, d is closely related to d_c , and properties 1–5 (see the beginning of §4.1) are satisfied.*

Unimodular quadratic forms also are natural candidates for the role of $\sigma_t \in \Sigma$. For example, given a θ -curve, consider a quadratic form Q_W on the lattice $\pi_1(T^2)$ (or rather on $H^1(T^2, \mathbb{Z})$, which is the same) that takes the vertices of the corresponding admissible hexagon W to 1. Then Q_W is a unimodular form with integer coefficients. Let us assign to $z \in H^2$ an ellipse E_z with semiaxes e^t and e^{-t} , where $t = d_H(z, i)$, such that the oriented angle between the Ox axis and the large axis of E_z is half the oriented angle between the geodesic lines going from i

²note that the center of a circle in H^2 usually does not coincide with its Euclidean center

to zero and to z . Let Q_z be a quadratic form defined by the condition $Q_z(\alpha) = 1$ if and only if $\alpha \in E_z$. Then $Q_W = Q_{z(W)}$, where $z(W)$ is the vertex of Γ that lies in the triangle Δ corresponding to an admissible hexagon W ; this can be verified by a straightforward calculation.

4.3. Other 3-dimensional manifolds.

4.3.1. Lens spaces. We have already discussed lens spaces in §1.3 and §2.4. Here we show that the distance function d_c coming from the Farey tessellation leads to an elegant reformulation of Conjecture 1. Note that d_c is well-defined for pairs of rational points on the absolute of H^2 and for pairs where one point belongs to H^2 and the other is a rational absolute point.

Lemma 12. *Let p, q be coprime positive integers. Then the Euclid complexity $E(p, q)$ defined in §1.2 is equal to $d_c(e^{2\pi i/3}, p/q)$.*

The point $e^{2\pi i/3} \in H^2$ can be replaced by any other inner point of the triangle $\Delta_0 = (\infty, -1, 0)$.

Proof. By Theorem 5, we have $d(W, W_0) = E(p, q)$, where the admissible hexagon W is uniquely defined by its leading vertex (p, q) (see [2] or §2.1 for definitions). Since $e^{2\pi i/3} \in \Delta_0$, we have $d_c(e^{2\pi i/3}, p/q) = d(W, W_0)$. \square

Recall that any lens space $L_{p,q}$ is obtained by gluing together two solid tori along their boundary torus T^2 . Denote by $\mu_1, \mu_2 \in \pi_1(T^2)$ the meridians (that is, contractible boundary cycles) of two solid tori. Fix some basis in $\pi_1(T^2) = \mathbb{Z}^2$. Then the slopes $r_1 = r(\mu_1), r_2 = r(\mu_2) \in \mathbb{Q}$ of the cycles μ_1 and μ_2 are defined.

Theorem 17. *We have $d_c(r_1, r_2) = E(p, q) - 1$.*

Compare this with Lemma 5 in §2.4.

Proof. The number $d_c(r(\mu_1), r(\mu_2))$ is independent of the choice of a basis in \mathbb{Z}^2 , because the action (4) of $\text{SL}(2, \mathbb{Z})$ on H^2 preserves the Farey tessellation and, consequently, the function d_c . For $L_{p,q}$, there exists a basis of \mathbb{Z}^2 such that $r_1 = 0$ and $r_2 = p/q$. Since $p > q > 0$, we have $p/q \in (1, \infty)$. Hence, the distance d_c of p/q to 0 is one less than its distance to an inner point of the triangle $(\infty, -1, 0)$, see Fig. 22. Now the statement of the theorem follows from Lemma 12. \square

Thus Conjecture 1 is equivalent to the relation $c(L_{p,q}) = d_c(r_1, r_2) - 2$.

Remark. For any two (and even three) absolute points $a, b \in \mathbb{R} \cup \infty$ there exists an isometry $A \in \text{SL}(2, \mathbb{R})$ of H^2 taking them, say, to 0 and 1. This is no longer the case for the $\text{SL}(2, \mathbb{Z})$ -action: obviously, only rational points (including ∞) can be mapped to rationals, but even a pair of rational numbers (r_1, r_2) cannot be mapped to the pair $(0, 1)$ whenever $d_c(r_1, r_2) > 0$. *Exercises:* 1) given two pairs of rational numbers, determine whether they are $\text{SL}(2, \mathbb{Z})$ -equivalent. (Hint: see [23]; the condition $d_c(r_1, r_2) = d_c(s_1, s_2)$ is necessary but not sufficient.) 2) Fix a basis in $\pi_1(T^2)$ and attach two solid tori along T^2 so that their meridians have slopes r_1 and r_2 . We get a manifold $M(r_1, r_2)$, which is a lens space. Show that $M(r_1, r_2)$ and $M(s_1, s_2)$ are homeomorphic if and only if unordered pairs (r_1, r_2) and (s_1, s_2) are $\text{SL}(2, \mathbb{Z})$ -equivalent. 3) Prove (once again) Theorem 3, see §1.3.

4.3.2. Stallings manifolds. Consider a fibration $p: M^3 \xrightarrow{F_g} S^1$, where F_g is an orientable surface (of genus g) and M^3 is an orientable 3-manifold. It may happen

that a manifold M can be fibered over a circle in several different ways, and the genus of the fiber need not be defined uniquely by M^3 . Choose any of the fibering structures and denote by \mathcal{A} the monodromy, which is an isotopy class of self-diffeomorphisms $A: F_g \rightarrow F_g$.

The ideas of Section 3 can be applied to this situation as well. Nothing changes in §3.1. Further, Lemma 7 remains true. Obviously, the set of isotopy classes of θ -curves should be replaced by the set of isotopy classes of trivalent graphs $L \subset F_g$ such that $F_g \setminus L$ is a 2-cell. With this correction, analogues of Lemma 8 and Theorem 13 hold, while the proof of Lemma 8 requires slight modification in the second paragraph. However, the difference between 1-skeletons L_- and L_+ of F_g in an analogue of Lemma 9 is measured by at most one Dehn twist (rather than at most one flip), which may require up to $4g - 3$ flips. So in Theorem 14 we only get the estimate $d(W_+, W_-) \leq 5(4g - 3)$, where W_+, W_- are isotopy classes of trivalent graphs embedded in F_g so that their complements are 2-cells, and d is the “flip-distance”. The remaining part of the reasoning of §3 goes smoothly, and we get an analogue of Theorem 12 with $1/5(4g - 3)$ instead of $1/5$. The approach discussed in §4.1 can be applied to this situation, too.

Another difference from the case of $g = 1$ is that we no longer know how to find $c(\mathcal{A})$, that is, the minimal number of flips required to convert a trivalent graph L such that $F_g \setminus L$ is a 2-cell into its monodromy image $\mathcal{A}L$. Recall that $c(\mathcal{A})$ can be computed by the methods of §2.2 or §4.2 whenever $g = 1$. The former approach is obstructed by the fact that the graph Γ is not a tree if $g > 1$. Probably, some estimates for $c(\mathcal{A})$ are easier to obtain than its exact value. In order to do it using the latter approach, one should replace the Farey tessellation of H^2 by the cell decomposition of the Teichmüller space of F_g described in [4].

The construction of a spine with small number of vertices presented in §2.3 gives a spine of M^3 with $c(\mathcal{A}) + 4g + 2$ vertices, and at least one of them may be cancelled by simplification moves provided that $c(\mathcal{A}) > 0$. It is not immediately clear how many vertices can be cancelled out of $4g + 2$. Anyway, it is plausible that any spine of M^3 contains more than $c(\mathcal{A})$ vertices, where \mathcal{A} is the monodromy of any possible fibering of M^3 over the circle.

4.3.3. Topological economy principle. If Conjecture 2 holds, a minimal spine of a torus bundle space $M(\mathcal{A})$ can be constructed as follows (see §2.3): fix a fiber $F = p^{-1}(0) \subset M^3$, then choose a θ -curve $L \subset F$ that requires $c(\mathcal{A})$ flips only to be converted to $\mathcal{A}L$, construct a simple polyhedron P_0 from the evolution of L , and, finally, add to P_0 an extra face that cuts any path connecting two boundary components of $M^3 \setminus F$ in the complement of P_0 .

According to Thurston’s geometrisation conjecture, any prime orientable compact 3-manifold M^3 can be cut into geometric pieces M_i , $i = 1, \dots, m$, along incompressible tori T_j^2 , $j = 1, \dots, n$, see, for example, the last section of [22]. For nontrivial cycles in the T_j^2 remain nontrivial in M^3 (by incompressibility), any spine P of M intersects all nontrivial cycles in any T_j^2 ; assuming general position, the intersections $P \cap T_j^2$ contain θ -curves for all j by virtue of Lemma 8.

Given a family of θ -curves $L_j \subset T_j^2$, consider simple polyhedra $P_i \subset M_i$ such that $P_i \cap T_j^2 = L_j$ for any boundary torus T_j^2 of M_i . Then add extra faces to P_i as necessary to get spines of the M_i . The union P of obtained polyhedra is a spine of M^3 (maybe, with several punctures). Minimize the total number of vertices of P over all possible choices of the P_i and the extra faces and over all “boundary

conditions”, that is, all families $L_j \subset T_j^2$. Needless to say, it is not clear yet how to implement this program in the general case.

Conjecture 5. *A minimal spine of any 3-manifold (that can be cut into geometric pieces) can be obtained by the procedure described in the paragraph above.*

In other words, there exists a minimal spine of M^3 that intersects any torus T_j^2 along one θ -curve only. Similar conjecture can be formulated about graph-manifolds [8, 30], which can be cut along incompressible tori into several copies of $D^2 \times S^1$ and $N^2 \times S^1$, where N^2 stands for D^2 with two holes. Conjecture 5 is yet another facet of the “topological economy principle” expressed and illustrated in [3].

REFERENCES

1. S. Anisov, *Flip equivalence of triangulations of surfaces*, (in Russian; English transl. in Moscow Univ. Math. Bull. **49** (1994) no. 2, 55–60), Vestnik Moskov. Univ. Ser. I Mat. Mekh. (1994), no. 2, 61–67.
2. S. Anisov and S. Lando, *Topological complexity of torus bundles over S^1* , in: Topics in Quantum Groups and Finite-type Invariants (Mathematics at the Independent University of Moscow) AMS Translations ser. 2, Vol. 185; Advances in the Mathematical Sciences (also available at <http://www.botik.ru/~duzhin/iuvvol1/anisov.ps.gz>) (B. Feigin, V. Vassiliev, eds.), 1998, pp. 129–135.
3. V. Arnold, *Topological problems in wave propagation theory and topological economy principle in algebraic geometry*, The Arnoldfest (Toronto, 1997), Fields Inst. Commun., **24**, (also available at <http://www.botik.ru/~duzhin/as/arnold/arnlect3.ps.gz>), Amer. Math. Soc., Providence, RI, 1999, pp. 39–54.
4. B. Bowditch and D. Epstein, *Natural triangulations associated to a surface*, Topology **27** (1988), 91–117.
5. B. Casler, *An embedding theorem for connected 3-manifolds with boundary*, Proc. Amer. Math. Soc. **16** (1965), 559–566.
6. A. Casson and S. Bleiler, *Automorphisms of Surfaces After Nielsen and Thurston*, London Mathematical Society Student Texts, 9, Cambridge University Press, Cambridge – New York, 1988, pp. iv+105.
7. H. Davenport, *The Higher Arithmetic. An Introduction to the Theory of Numbers*, Cambridge University Press, Cambridge, UK, 1992, pp. 217.
8. A. Fomenko and S. Matveev, *Algorithmic and computer methods in three-dimensional topology*, (in Russian; English transl.: *Algorithmic and computer methods for three-manifolds*. Mathematics and its Applications, No. 425, Kluwer Academic Publishers, Dordrecht, 1997, pp. xii+334), Moskov. Gos. Univ., Moscow, 1991, pp. 303.
9. M. Goresky and R. MacPherson, *Stratified Morse Theory*, Results in Mathematics and Related Areas (3), 14, Springer–Verlag, Berlin–New York, 1988, pp. xiv+272.
10. M. Gromov, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. **56** (1982), 5–99.
11. G. Hardy and E. Wright, *An introduction to the theory of numbers*, the Clarendon Press, Oxford – New York, 1979, pp. xvi+426.
12. S. Matveev, *Transformations of special spines, and the Zeeman conjecture*, (in Russian; English transl. in Math. USSR-Izv. 31 (1988), no. 2, 423–434), Izv. Akad. Nauk SSSR Ser. Mat. **51** (1987), no. 5, 1104–1116, 1119.
13. S. Matveev, *Computer recognition of three-manifolds*, Experimental Mathematics **7** (1998), no. 2, 153–161.
14. S. Matveev, *Tables of 3-manifolds up to complexity 6*, (.dvi and .ps files are available through <http://www.mpim-bonn.mpg.de/html/preprints/preprints.html> ; the .ps file exceeds 60 Mbytes), Max Planck Institute preprint MPI 1998-67, 1–50.
15. S. Matveev, *Complexity theory of three-dimensional manifolds*, Acta Appl. Math. **19** (1990), 101–130.
16. J. Milnor, *Morse Theory*, Annals of Mathematics Studies, No. 51, Princeton University Press, Princeton, N.J., 1963, pp. vi+153.

17. J. Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, No. 61, Princeton University Press, Princeton, N.J., 1968, pp. iii+122.
18. V. Nikulin and I. Shafarevich, *Geometries and Groups*, (in Russian; English transl.: *Geometries and Groups*, Universitext. Springer Series in Soviet Mathematics, Springer-Verlag, Berlin-New York, 1987, pp. viii+251), Nauka, Moscow, 1983, pp. 240.
19. M. Ovchinnikov, *A table of closed orientable irreducible 3-manifolds of complexity 7*, (in Russian), Chelyabinsk University preprint (1997).
20. U. Pachner, *Bistellare Äquivalenz kombinatorischer Mannigfaltigkeiten*, (in German), Arch. Math. (Basel) **30** (1978), no. 1, 89–98.
21. E. Pervova, *Morse functions on simple polyhedra, and related construction of special spines of manifolds*, to appear in 2001 (in Russian), in: *Geometry and Applications*, proceedings of the V. Toponogov 70th birthday conference, Novosibirsk, 2000.
22. P. Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. **15** (1983), no. 5, 401–487.
23. C. Series, *The modular surface and continued fractions*, J. London Math. Soc. (2) **31** (1985), no. 1, 69–80.
24. J.-P. Serre, *Cours d'arithmétique*, (in French; English transl.: *A course in arithmetic*. Graduate Texts in Mathematics, No. 7, Springer-Verlag, New York-Heidelberg, 1973, pp. viii+115), Collection SUP: “Le Mathématicien”, 2, Presses Universitaires de France, Paris, 1977, pp. 188.
25. J.-P. Serre, *Arbres, amalgames, SL_2* , (in French; English transl.: *Trees*, Springer-Verlag, Berlin-New York, 1980, pp. ix+142), Astérisque, No. 46, Société Mathématique de France, Paris, 1977, pp. 189.
26. D. Sleator, R. Tarjan, and W. Thurston, *Rotation distance, triangulations, and hyperbolic geometry*, Journal of the Amer. Math. Soc. **1** (1988), no. 3, 647–681.
27. M. Takahashi and M. Ochiai, *Heegard diagrams of torus bundles over S^1* , Comment. Math. Univ. St. Paul. **31** (1982), no. 1, 63–69.
28. V. Turaev and O. Viro, *State sum invariants of 3-manifolds and quantum 6j-symbols*, Topology **31** (1992), no. 4, 865–902.
29. I. Vinogradov, *Foundations of Number Theory*, (in Russian; English transl.: *An Introduction to the Theory of Numbers*. Pergamon Press, London-New York, 1955, pp. vi+155), Nauka, Moscow, 1981.
30. F. Waldhausen, *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten. I, II*, (in German), Invent. Math. **3** (1967), 308–333 and **4** (1967), 87–117.

INSTITUTE FOR SYSTEM STUDIES, RUSSIAN ACADEMY OF SCIENCES

Current address: IHES, Le Bois-Marie, 35, Route de Chartres, F-91440 Bures-sur-Yvette, France

E-mail address: anisov@mccme.ru